

Convex Feasibility via Monotropic Programming

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Outline

- 1 Problem Formulation
- 2 Monotropic Programming
- 3 Preliminaries
- 4 Facts
- 5 Analysis of Consistency

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The problem formulation

Let H be a Hilbert space and let $C_n, n = 1, \dots, m$ be convex closed subsets of H . The **convex feasibility problem** is to find some point

$$x \in \bigcap_{n=1}^m C_n \quad (CFP)$$

when this intersection is non-empty.

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The *CFP* has wide ranging applications:

- medical imaging, computerised tomography, signal processing.
- Partial differential equations (Dirichlet problem), complex analysis (Bergman kernels, conformal mappings);
- Subgradient algorithms with application in solution of convex inequalities, minimization of convex nonsmooth functions.

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Fact (Bauschke-Borwein, 1996)

- CFP equivalent to problem involving only two convex and closed sets in $H^m = H \times \dots \times H$ consisting of m copies of H , with the additional advantage that one of these sets is a linear subspace
- Hence, from now on we assume that we are dealing with only two (possibly disjoint) closed convex sets.

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Monotropic Model (Minty, 1960) (Rockafellar, 1970, 1981, 1998)

$$\min \sum_{i=1}^m f_i(x_i) \quad (P)$$

subject to $(x_1, \dots, x_m) \in S$,

- $f_i : H_i \rightarrow \mathbb{R} \cup \{+\infty\}$ proper, convex,
- $S \subseteq \prod_{i=1}^m H_i$ is a closed linear subspace

(P) will be our primal model.

(P) has a very symmetric dual problem:

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Dual of (P)

$$\max \sum_{i=1}^m -f_i^*(x_i^*) \quad (D)$$

subject to $(x_1^*, \dots, x_m^*) \in S^\perp$,

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Basic Ingredients:

- The *Fenchel conjugate* of f is $f^* : H \rightarrow \mathbb{R} \cup \{+\infty\}$

$$f^*(v) := \sup_{x \in H} \{\langle v, x \rangle - f(x)\}$$

- The *subdifferential* of f at x is defined by

$$\partial f(x) := \{v \in H \mid \langle v, y - x \rangle \leq f(y) - f(x), \text{ for all } y \in H\},$$

if $f(x) \in \mathbb{R}$, and \emptyset otherwise.

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Basic Ingredients (II):

- For $C \subset H$, the *indicator function* of C is $\iota_C(x) := 0$ if $x \in C$ and $\iota_C(x) := +\infty$ otherwise.
- The the *support function* of C is

$$\sigma_C(v) := \sup_{y \in C} \langle v, y \rangle$$

for $v \in H$

Easy to check

$$(\iota_C)^* = \sigma_C$$



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Basic Ingredients (III):

For $\psi_1, \psi_2 : H \rightarrow \mathbb{R} \cup \{+\infty\}$, their *infimal convolution* is defined by

$$(\psi_1 \square \psi_2)(z) := \inf_{z_1+z_2=z} \{\psi_1(z_1) + \psi_2(z_2)\}.$$

For $f : H \rightarrow \mathbb{R} \cup \{+\infty\}$ recall that the *epigraph* is the set

$$\text{epi } f := \{(x, r) \in H \times \mathbb{R} : f(x) \leq r\}$$

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Fact (B.-Jeyakumar, 2005):

$C, D \subset H$ closed convex:

$$C \cap D \neq \emptyset \iff (0, -1) \notin \text{cl}(\text{epi } \sigma_C + \text{epi } \sigma_D)$$

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Primal for CFP:

Our problem is (recall we reduced the problem to 2 sets):

$$\text{find } (x, y) \in C_1 \times C_2 \subset H \times H, \text{ such that } x = y$$

which can be formulated as

$$\min_{(x,y) \in S} d_{C_1}(x) + d_{C_2}(y) \quad (P)$$

where $S = \{(x, y) \in H^2 : x = y\}$.

Dual for CFP:

Using monotropic formulation we obtain its dual:

$$\sup_{(v,w) \in S^\perp} -d_{C_1}^*(v) - d_{C_2}^*(w) \quad (D)$$

where $S^\perp = \{(u, v) \in H^2 : u + v = 0\}$.

What do we know about this primal-dual pair?

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Duality facts:

Pro 15.22 and Theo 19.1 from Bauschke-Combettes book yield:

$$v(P) = v(D) \text{ and } (D) \text{ always has a solution}$$

In this situation, (x, y) solves (P) and (u, v) solves (D) .
Moreover,



$$\begin{array}{ll} (x, y) \in S, & (u, v) \in S^\perp \\ u \in \partial d_{C_1}(x) & v \in \partial d_{C_2}(y) \end{array}$$

Proof not very direct!

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$$\begin{aligned} \sup_{v \in H} -d_{C_1}^*(v) - d_{C_2}^*(-v) &= \text{✍} \\ &= -\min_{t \in [0,1]} t \underbrace{\left(\inf_{\|v\| \leq 1} \sigma_{C_1}(v) + \sigma_{C_2}(-v) \right)}_{\Phi(1)}, \end{aligned}$$

which gives an equivalent reformulation of the dual in terms of $\Phi(1)$. Always $\Phi(1) \leq 0$. Value $\Phi(1)$ gives important information:

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Consistency results for CFP:

1. $\Phi(1) < 0 \iff 0 \notin \text{cl}(C_2 - C_1)$. So $C_1 \cap C_2 = \emptyset$.
2. $\Phi(1) = 0 \iff 0 \in \text{cl}(C_2 - C_1)$. This leads to two cases:

2.1 If $(\sigma_{C_1} \square \sigma_{C_2})$ is lsc at 0, then $C_1 \cap C_2 \neq \emptyset$.

$$\text{and } 0 \in (C_2 - C_1)$$

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 - If $(\sigma_{C_1} \square \sigma_{C_2})$ is not lsc at 0 then $C_1 \cap C_2 = \emptyset$, \exists (possibly improper) closed separating hyperplane.
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Characterization of Consistency:

Assume that $(\sigma_{C_1} \square \sigma_{C_2})(0) > -\infty$. Then $(\sigma_{C_1} \square \sigma_{C_2})$ is proper, and TFSAE:

(i) $C_1 \cap C_2 \neq \emptyset$,

(ii) $(\sigma_{C_1} \square \sigma_{C_2})$ is lsc at 0,

(iii) $\{0\} \times \mathbb{R} \cap \text{epi}(\sigma_{C_1} \square \sigma_{C_2}) = \{0\} \times \mathbb{R}_+$

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Consistency for CFP in the critical case $v(D) = 0$:

Recall that (D) always has solutions. Assume $v(D) = 0$. Then:

- (a) If $v = 0$ is unique solution of $(D) \iff C_1 \cap C_2 \neq \emptyset$.
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Inconsistency for CFP in critical case $d(C_1, C_2) = 0$. TFSAE:

- (i) (P) has no solutions.
- (ii) $0 \in \text{cl}(C_1 - C_2) \setminus (C_1 - C_2)$.
- (iii) $\sigma_{C_1} \square \sigma_{C_2}$ is not lsc at 0.
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