Regularity of mappings vs transversality of collections of sets

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Regularity/transversality

- Constraint qualifications
- Qualification conditions in subdifferential calculus
- Qualification conditions in convergence analysis

- Banach open mapping principle
- Lyusternik–Graves theorem

- Transversality of sets in differential geometry
- Separation theorem

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Regularity of (multi)functions	$\Longleftrightarrow$	Transversality of collections of sets

Outline

1. Transversality/subtransversality

2. Pairs of sets vs set-valued mappings

3. Dual characterizations of transversality/subtransversality

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Subtransversality

X – normed vector space, $A, B \subset X$, $\bar{x} \in A \cap B$

Definition

$\{A, B\}$ is subtransversal at $\bar{x}$ if $\exists \alpha, \delta > 0$ such that

$$\alpha d\,(x, A \cap B) \leq \max\{d(x, A), d(x, B)\} \quad \forall x \in \mathbb{B}_\delta(\bar{x})$$

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(Dolecki, 1982); (Ioffe, 1989); (Local) linear regularity (Bauschke, Borwein, 1993); linear estimate, linear coherence (Penot, 1998, 2013); metric inequality (Ngai, Théra, 2001); subtransversality (Ioffe, 2015)

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$\mathrm{str}[A, B](\bar{x}) := \sup\{\alpha \text{ in the above condition}\}$

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$\forall x \in \mathbb{B}_\delta(\bar{x}),\ x_1, x_2 \in \delta\mathbb{B}$

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Strong metric inequality (K., 2005)

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$\mathrm{tr}[A, B](\bar{x}) := \sup\{\alpha$ in the above definition$\}$

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$0 \leq \mathrm{tr}[A, B](\bar{x}) \leq \mathrm{str}[A, B](\bar{x})$

Transversality $\implies$ Subtransversality

Outline

Metric (sub-)regularity

X, Y – metric spaces, $\quad F : X \rightrightarrows Y, \quad (\bar{x}, \bar{y}) \in \operatorname{gph} F$

Definition

F is metrically regular at $(\bar{x}, \bar{y})$ if $\exists \alpha, \delta > 0$ such that

$$\alpha d\left(x, F^{-1}(y)\right) \le d(y, F(x)) \quad \forall x \in \mathbb{B}_\delta(\bar{x}),\ y \in \mathbb{B}_\delta(\bar{y})$$

$\mathrm{rg}[F](\bar{x}, \bar{y}) = \sup\{\alpha$ in the above definition$\}$

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Definition

F is metrically subregular at $(\bar{x}, \bar{y})$ if $\exists \alpha, \delta > 0$ such that

$$\alpha d\left(x, F^{-1}(\bar{y})\right) \le d(\bar{y}, F(x)) \quad \forall x \in \mathbb{B}_\delta(\bar{x})$$

$\mathrm{srg}[F](\bar{x}, \bar{y}) = \sup\{\alpha$ in the above definition$\}$

Pairs of sets vs set-valued mappings

X – normed vector space, $A, B \subset X$, $\bar{x} \in A \cap B$

$F : X \rightrightarrows X^2$: $F(x) := (A - x) \times (B - x)$ (Ioffe, 2000)

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Proposition (Ioffe, 2000; K., 2005)

1. $\{A, B\}$ *is transversal at* $\bar{x}$ $\iff$ F *is metrically regular at* $(\bar{x}, 0)$
2. $\{A, B\}$ *is subtransversal at* $\bar{x}$ $\iff$ F *is metrically subregular at* $(\bar{x}, 0)$

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$\mathrm{tr}[A, B](\bar{x}) = \mathrm{rg}[F](\bar{x}, 0)$ $\qquad$ $\mathrm{str}[A, B](\bar{x}) = \mathrm{srg}[F](\bar{x}, 0)$

Collections of sets vs set-valued mappings

X, Y – normed vector spaces, $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$

$A = \operatorname{gph} F$, $B = X \times \{\bar{y}\}$

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Proposition (K., 2005)

1. *F is metrically regular at $(\bar{x}, \bar{y})$ $\iff$ $\{A, B\}$ is transversal at $(\bar{x}, \bar{y})$*
2. *F is metrically subregular at $(\bar{x}, \bar{y})$ $\iff$ $\{A, B\}$ is subtransversal at $(\bar{x}, \bar{y})$*

Collections of sets vs set-valued mappings

X – Euclidean space, $\quad A, B \subset X, \quad \bar{x} \in A \cap B$

$\Phi : X^2 \rightrightarrows X$: $\quad \Phi(x_1, x_2) := \begin{cases} \{x_1 - x_2\} & \text{if } x_1 \in A \text{ and } x_2 \in B \\ \emptyset & \text{otherwise} \end{cases}$

(Lewis & Malick, 2008)

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Proposition

1. $\{A, B\}$ *is transversal at* $\bar{x} \iff \Phi$ *is metrically regular at* $((\bar{x}, \bar{x}), 0)$
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3. Dual characterizations of transversality/subtransversality

Dual characterizations: normals

X – normed vector space, $A \subset X$, $\bar{x} \in A$

Fréchet normal cone to A at $\bar{x}$:

$$N_A(\bar{x}) := \left\{ x^* \in X^* \mid \limsup_{a \to \bar{x},\, a \in A \setminus \{\bar{x}\}} \frac{\langle x^*, a - \bar{x} \rangle}{\|a - \bar{x}\|} \le 0 \right\}$$

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$\dim X < \infty$

Limiting normal cone to A at $\bar{x}$:

$$\overline{N}_A(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{a \to \bar{x},\, a \in A} N_A(a)$$

$$:= \left\{ x^* = \lim_{k \to \infty} x_k^* \mid x_k^* \in N_A(a_k),\ a_k \in A,\ a_k \to \bar{x} \right\}$$

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X – Euclidean space, A – closed

Proximal normal cone to A at $\bar{x}$:

$$N_A^p(\bar{x}) := \operatorname{cone}\left(P_A^{-1}(\bar{x}) - \bar{x}\right)$$

Dual characterizations: normals

X – Euclidean space, A – closed, $\bar{x} \in A$

$$N_A(\bar{x}) = \left\{ v \in X \mid \limsup_{a \to \bar{x},\, a \in A \setminus \{\bar{x}\}} \frac{\langle v, a - \bar{x} \rangle}{\|a - \bar{x}\|} \leq 0 \right\}$$

$$N_A^p(\bar{x}) = \operatorname{cone} \left(P_A^{-1}(\bar{x}) - \bar{x} \right)$$

$$\overline{N}_A(\bar{x}) = \operatorname*{Lim\,sup}_{a \to \bar{x},\, a \in A} N_A^p(a) = \operatorname*{Lim\,sup}_{a \to \bar{x},\, a \in A} N_A(a)$$

$$N_A^p(\bar{x}) \subseteq N_A(\bar{x}) \subseteq \overline{N}_A(\bar{x})$$

Dual characterizations: transversality

X – Asplund space, A, B – closed, $\bar{x} \in A \cap B$

Theorem (K., 2005)

$\{A, B\}$ *is transversal at* $\bar{x}$ $\iff$ $\exists \alpha, \delta > 0$ *such that* $\|x_1^* + x_2^*\| > \alpha$ $\forall a \in A \cap \mathbb{B}_\delta(\bar{x})$, $b \in B \cap \mathbb{B}_\delta(\bar{x})$, $x_1^* \in N_A(a)$, $x_2^* \in N_B(b)$, *satisfying* $\|x_1^*\| + \|x_2^*\| = 1$

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$\mathrm{tr}[A, B](\bar{x}) = \sup\{\alpha$ in the above condition$\}$

Dual characterizations: transversality

$\dim X < \infty, \quad A, B$ – closed, $\quad \bar{x} \in A \cap B$

$\{A, B\}$ is transversal at $\bar{x} \quad \Longleftrightarrow \quad \exists \alpha > 0$ such that $\|x_1^* + x_2^*\| > \alpha$ $\forall x_1^* \in \overline{N}_A(\bar{x}),\ x_2^* \in \overline{N}_B(\bar{x})$, satisfying $\|x_1^*\| + \|x_2^*\| = 1$

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$\{A, B\}$ is transversal at $\bar{x}$ $\iff$ $\overline{N}_A(\bar{x}) \cap (-\overline{N}_B(\bar{x})) = \{0\}$

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$\{A, B\}$ is transversal at $\bar{x}$ $\iff$ $\exists \alpha > 0$ such that $\|x_1^* + x_2^*\| > \alpha$ $\forall x_1^* \in \overline{N}_A(\bar{x})$, $x_2^* \in \overline{N}_B(\bar{x})$, satisfying $\|x_1^*\| + \|x_2^*\| = 1$

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Transversality (Clarke et al, 1998); normal qualification condition (Mordukhovich, 2006; Penot, 2013); regular intersection (Lewis & Malick, 2008); linearly regular intersection (Lewis et al, 2009); alliedness property (Penot, 2013); transversal intersection (Ioffe, 2015) ...

Dual characterizations: subtransversality

X – Asplund space, A, B – closed, $\bar{x} \in A \cap B$

Theorem (K., Luke, Thao, 2017)

$\{A, B\}$ is subtransversal at $\bar{x}$ if $\exists \alpha, \delta > 0$ such that

$$\|x_1^* + x_2^*\| > \alpha$$

$\forall x \in X$, $a \in A \setminus B$, $b \in B \setminus A$ with $\|x - \bar{x}\| < \delta$, $0 < \|x - a\| < \delta$, $0 < \|x - b\| < \delta$, and all nonzero $x_1^* \in N_A(a)$, $x_2^* \in N_B(b)$ satisfying

$$\|x_1^*\| + \|x_2^*\| = 1, \quad \frac{\langle x_1^*, x - a\rangle}{\|x_1^*\|\,\|x - a\|} > 1 - \delta, \quad \frac{\langle x_2^*, x - b\rangle}{\|x_2^*\|\,\|x - b\|} > 1 - \delta$$

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Theorem (K., Luke, Thao, 2017)

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$\mathrm{str}[A, B](\bar{x}) \geq \sup\{\alpha \text{ in the above theorem}\}$

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Thank you