Open problems in convex optimisation

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AMSI Optimise

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Perceptron algorithm and its complexity

Find an $x \in \mathbb{R}^n$ such that

$$a_i^T x > 0 \quad \forall i \in \{1, \dots, m\},$$

where $a_i \in \mathbb{R}^n$, and we assume in addition that $\|a_i\| = 1$ for all i.

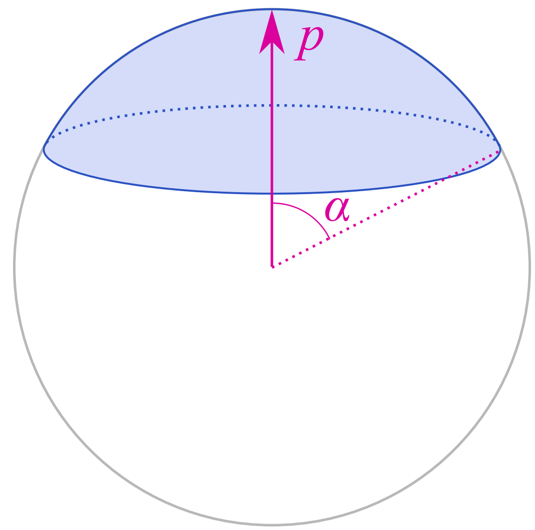
The perceptron algorithm

1. Let $x_0 := 0$, $k = 0$
2. Choose i such that $a_i^T x_k$ is minimal
3. Let $x_{k+1} := x_k + a_i$.
4. If $a_i^T x_{k+1} > 0 \, \forall i$, halt. Otherwise, $k := k + 1$ and go to 2.

If the problem is feasible, the perceptron algorithm terminates in finitely many steps, and the number of steps depends on 'geometry' of the problem.

Spherical caps and complexity

For a given $p \in \mathcal{S}^{n-1}$ and $\alpha \in [0, \pi]$ the *spherical cap* in $\mathcal{S}^{n-1}$ with centre at p and angular radius α is defined as



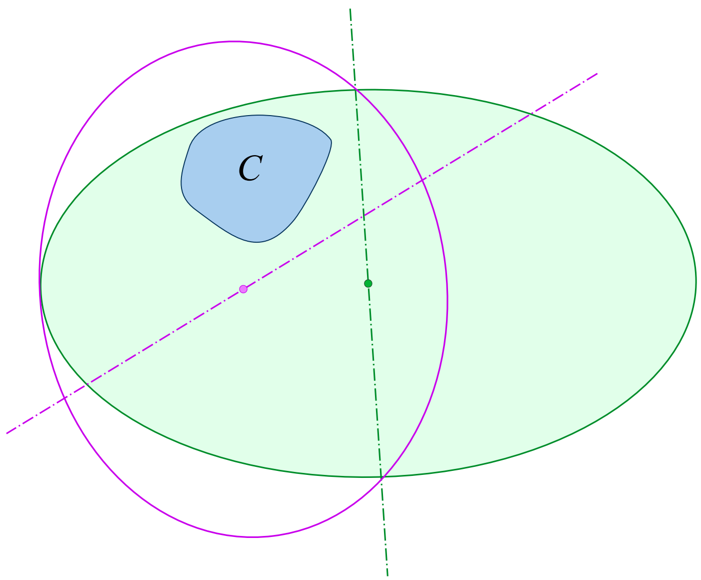
$$\text{cap}(p, \alpha) := \{x \in \mathcal{S}^{n-1} \,|\, p^T x \geq \cos \alpha\}.$$

A *smallest including spherical cap* for $a_1, \ldots, a_m \in \mathcal{S}^{n-1}$ is such that the angle α is the smallest possible.

The quantity $\cos \alpha$ can be used to measures the 'difficulty' of the problem $a_i^T x > 0$. If $\cos \alpha > 0$ (and hence the problem has a solution), the perceptron algorithm halts in $\frac{1}{\cos^2 \alpha}$ iterations.

[P. Bürgisser and F. Cucker *Condition*, 2013]

Ellipsoid method



The ellipsoid method finds an interior point of a compact convex set $C \subset \mathbb{R}^n$ using a *separation oracle*.

The algorithm starts with a ball $B(y_0, R)$ of radius R centred at $y_0 \in \mathbb{R}^n$ such that $C \subset B(y_0, R)$.

0. Let $k = 0$, $E_0 = B(y_0, R)$.
1. If $y_k \in C$, output y_k and halt, otherwise find a separating half-space H and compute the minimal ellipsoid $E_{k+1} \supseteq E_k \cap H$.
2. Let y_{k+1} be the centre of E_{k+1}, set $k = k+1$ and go to step 1.

Ellipsoid method

The number of iterations is bounded by

$$\left\lceil 2n \ln \frac{\mathrm{vol}(B(y_0, R))}{\mathrm{vol}\, C} \right\rceil.$$

[L. Tunçel, *Polyhedral and semidefinite programming methods in combinatorial optimization*, 2010]

Similar bounds hold for a pair of feasibility problems

$$\begin{array}{rcl} Ax & = & 0 \\ x & \in & K \end{array} \quad (\mathrm{P}) \qquad A^T y \in K, \quad (\mathrm{D})$$

where K is a symmetric cone, decided by an ipm in

$$O(\sqrt{\nu_K} \ln(\nu_K C_R(A))) \quad \text{iterations.}$$

[D. Amelunxen, P. Bürgisser, Intrinsic volumes of symmetric cones...]

Smale's 9^{th} problem

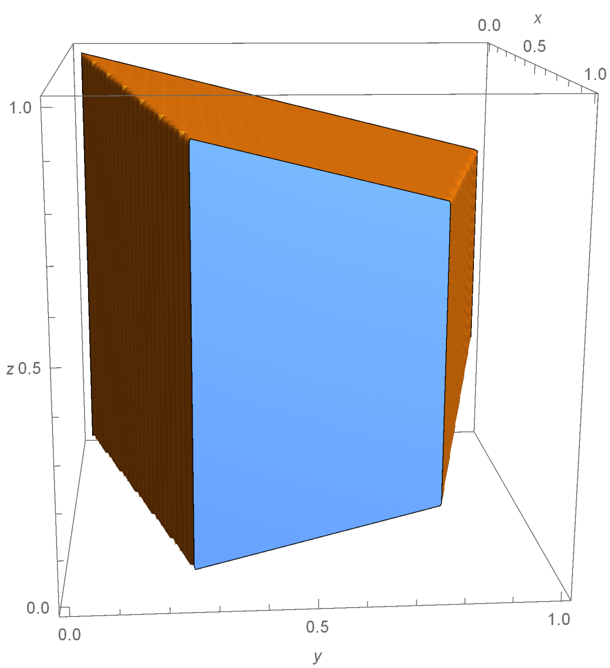
> Is there a polynomial time algorithm over the real numbers which decides the feasibility of the linear system of inequalities $Ax \geq b$?

Here the key is 'over the real numbers': this means that the algorithm should terminate after polynomially many algebraic operations. This polynomial bound should be in terms of the dimension of the problem.

Simplex method and the Klee-Minty cube

Simplex method solves a problem of maximising a linear objective over a polytope defined by a system of inequalities by travelling from one vertex to another and hence increasing the value of the objective function until the optimal solution is reached.

Klee-Minty cube refers to a range of examples that demonstrate the exponential inefficiency of the simplex method.



$$
\begin{aligned}
\text{min} \quad & x_n \\
& 0 \le x_1 \le 1, \\
& \varepsilon x_{i-1} \le x_i \le 1 - \varepsilon x_{i-1},\ i \in 2:n.
\end{aligned}
$$

[Gärtner, Henk, Ziegler, Randomized simplex algorithms...]

Polynomial Hirsch conjecture

The Hirsch conjecture stated that for any d-dimensional polytope that is bounded by n inequalities, the shortest path between any two vertices is bounded by $n - d$.

This conjecture was disproved by Franciso Santos in 2010. He constructed a counterexample in 43 dimensional space of a polytope with 86 facets and combinatorial diameter at least 44.

The quest now is to solve the 'polynomial Hirsch conjecture', i.e. to find out if there is a polynomial bound on the diameter of polytopes in terms of the number of facets and dimension. The best known bounds are as follows (there are better bounds of the same order):

$$\Delta(n,d) \leq (n-d)^{\log d} \qquad \text{and} \qquad \Delta(n,d) \leq \frac{2^{d-2}}{3} n.$$

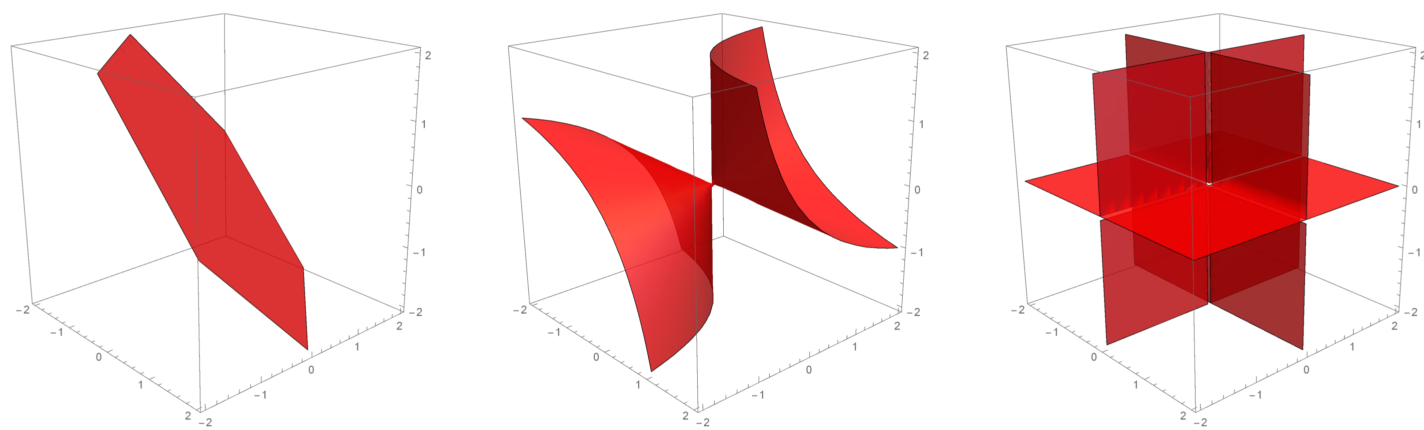
[F. Santos, A counterexample to the Hirsch conjecture]

Hyperbolicity cones

A homogeneous polynomial $p \in \mathbb{R}[x_1, x_2, \dots, x_n]$ is called *hyperbolic* in a direction $e \in \mathbb{R}^n$ if $p(e) \neq 0$ and the univariate polynomial $t \mapsto p(x + te)$ has only real roots for all x.

A hyperbolicity cone is the connected component of the complement of $\{x \,|\, p(x) = 0\}$ which contains e.

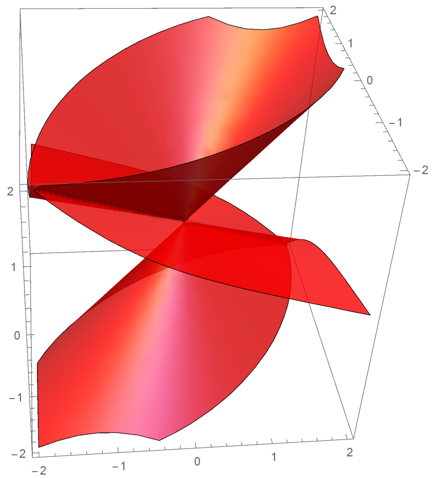
Examples of the set $\{x \,|\, p(x) = 0\}$ for elementary symmetric polynomials $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$ and $x_1x_2x_3$:



Hyperbolicity cones

The polynomial $p(x, y, z) = 4xyz + xz^2 + yz^2 + 2z^3 - x^3 - 3zx^2 - y^3 - 3zy^2$ is hyperbolic with respect to $(0, 0, 1)$.

[see Pablo Parrilo's notes *Algebraic techniques and semidefinite optimization*]



The cones $\mathbb{R}^n_{++}$ and $\mathbb{S}^n_{++}$ are hyperbolicity cones.

The generalised Lax conjecture states that a hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices.

The conjecture is true in $\mathbb{R}^3$. Many special cases were proven recently.

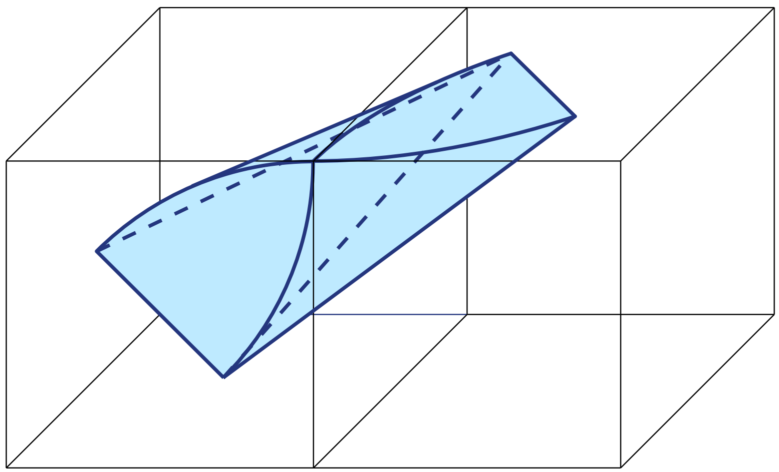
[Helton, Vinnikov, Linear matrix inequality representation of sets, 2007]

Some problems and (partial) solutions

Facial Dual Completeness (FDC) property is useful for a range of applications, including the facial reduction algorithm.

[Pataki, On the connection of facially exposed and nice cones 2012]

A closed convex cone $K \subset \mathbb{R}^n$ is facially dual complete if for every face $F \lhd K$ the sum $F^{\perp} + K^*$ is closed.



FDC coincides with facial exposure in $\mathbb{R}^3$, but in higher dimensions the two notions are essentially different.

$\leftarrow$ a slice of a cone that is facially exposed but not FDC.

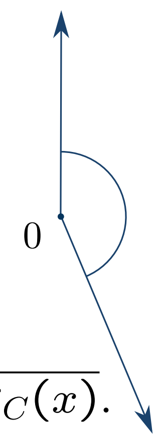
[R. Facially exposed cones are not always nice, 2014]

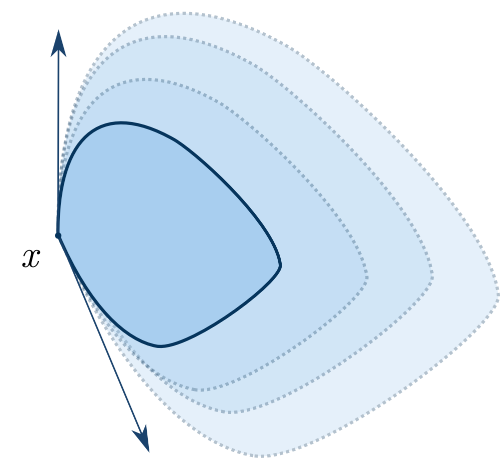
Primal characterisations of FDC

Theorem 1. *If a closed convex cone K is facially dual complete, then for every $F \lhd K$ and every $x \in F$, we have*

$$T_K(x) \cap \mathrm{span} F = T_F(x). \tag{1}$$

Here $T_C(x)$ is the tangent cone to the set C at a point $x \in C$. A tangent cone can be defined as the closure of the cone of feasible directions:





$$\mathrm{dir}_C(x) = \bigcup_{\lambda \in \mathbb{R}_+} \lambda(C - x), \qquad T_C(x) = \overline{\mathrm{dir}_C(x)}.$$

Equivalently

$$T_C(x) = \{y \in \mathbb{R}^n \,|\, \exists \{z_k\}, \{t_k\} : z_k + x \in C, t_k \geq 0 \,\forall\, k, t_k z_k \to y\}.$$

Sufficient condition

Theorem 2 (Sufficient condition)**.** *If a closed convex cone* $K \subset \mathbb{R}^n$ *is such that the condition*

$$T_K(x) \cap \text{span}F = T_F(x) \quad \forall F \lhd K, \forall x \in F.$$

holds not only for the cone K*, but for all lexicographic tangents (tangent of a tangent of a tangent...), then* K *is Facially Dual Complete.*

[R., Tunçel, Facially Dual Complete cones and lexicographic tangents 2017]

Chains of faces and dimensions

Each face of the cone of positive semidefinite matrices is linearly isomorphic to a lower dimensional SDP cone. Any nested chain of faces of $\mathbb{S}^n_+$ has length no more than n.

The SDP cone is a peculiar example of a convex set that has very large 'gaps' in the dimensions of its faces.

Theorem 3. *For any finite increasing sequence of integers* $0 < d_1 < d_2 < \cdots < d_k$ *there exists a compact convex set* C *in* $\mathbb{R}^{d_k}$ *such that the faces of this set come in dimensions that form this sequence.*

[R, Sang, Yost, *On the dimensions of faces of compact convex sets*]

Question: is it possible to obtain every sequence from affine slices of the SDP cone?

Thank you