Automatic Logic-based Benders Decomposition with MiniZinc

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Outline

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Multi-resource Scheduling

$$\begin{aligned}\textbf{minimize} \quad & \sum_{r \in R} \text{cost of schedule for } r \\ \textbf{s.t.} \quad & \forall_{t \in T}.\ \text{task } t \text{ is scheduled on some } r \\ & \forall_{r \in R}.\ \text{schedule for } r \text{ is feasible}\end{aligned}$$

Frequently:

- Objective is a linear combination of 0–1 variables
- Feasibility constraint is something nastily combinatorial
  - Cumulative resource capacities, bin packing, . . .

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Integer Programming?

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Constraint Programming?

- Specialized reasoning for many combinatorial constraints
- Much weaker bounding than MIP.
    - Only tightens objective bounds when defining variables change.

Logic-Based Benders Decomposition

Decompose the problem into a (MIP) master problem over shared variables, and several independent (CP) subproblems.

Master: Assign tasks to machines

Subproblem: Schedule tasks on a single machine

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- Find an optimal solution μ to the master
- Search for a feasible extension of μ to each subproblem
  - If all subproblems are feasible, we have found an optimum.
  - Otherwise, add a cut to the master and restart.

In theory, cuts are derived by solving the inference dual.
In practice, some form of generate-and-test.

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$$\boxed{M}$$

$$\boxed{S_1} \quad \boxed{S_2} \quad \boxed{S_3}$$

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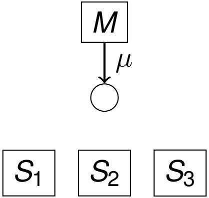

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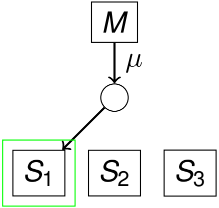

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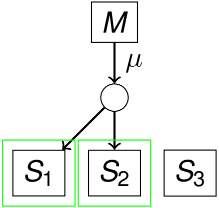

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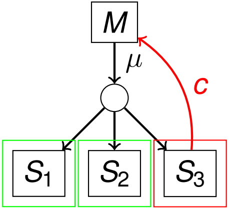

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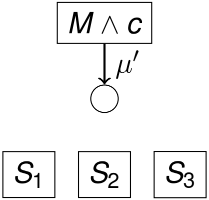

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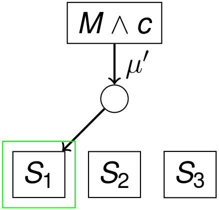

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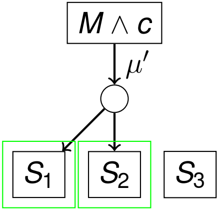

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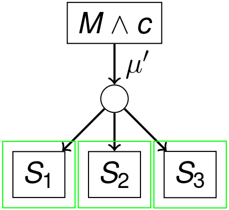

Automating Logic-based Benders Decomposition

An effective strategy, but sees surprisingly little use.

- Specialized implementation per-problem.
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Limitations to be aware of:

- Frequently all-or-nothing (optimal solution or none)
- Subproblems must be fully independent (no coupling)

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What elements do we need for automating LBBD?

1. Automatic partitioning into master/subproblems
2. Systematic extraction of cuts from arbitrary subproblems

Outline

MiniZinc: A solver-independent modelling language

- solver-independent
  - supported by CP, MIP, SAT, SMT, and local search solvers
- high-level
  - encode combinatorial substructures directly as global constraints
- defacto standard for CP modelling

Hands On Session

Learn MiniZinc: Wednesday 28th: 11:00 - 12:30

MiniZinc: How it works

MiniZinc High-level model specification translates to ...

```
constraint forall (m in machines) (
  cumulative(
    [starts[j] | j in jobs],
    [duration[j,m] | j in jobs],
    [resource[j,m]*bool2int(assign[j] = m) | j in
        jobs],
    capacities[m]
  )
);
```

FlatZinc Variable declarations and primitive constraints

```
% ...
constraint cumulative(X_INTRODUCED_233,
    X_INTRODUCED_235,X_INTRODUCED_234,15);
constraint int_lin_le([1,-1],[X_INTRODUCED_27,
    objective],-6);
% ...
```

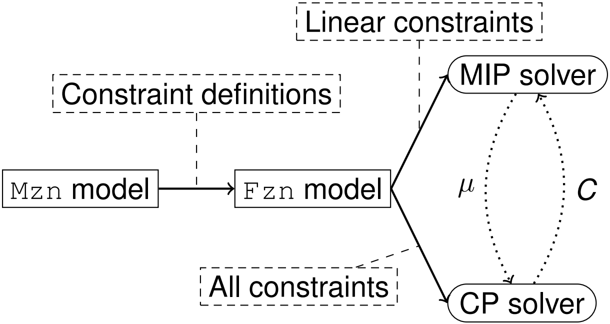
Flattening uses a solver-specific library of transformations.

Outline

Automating 'decomposition'

A simple strategy: MIP master, single CP subproblem.

- Master contains all linear inequalities (and corresponding variables).
- Subproblem contains everything (as if solving directly with CP).



Implicit subproblems

With classical CP, this is a terrible idea.

$$P = \begin{array}{l} p_1 + p_2 + p_3 \leq 2 \\ \wedge\ x_1 + x_2 \leq p_1 \\ \wedge\ y_1 + y_2 \leq p_2 \\ \wedge\ z_1 = z_2 + p_3 \wedge z_1 \neq z_2 \end{array}$$

Assuming $\{p_1 = 1, p_2 = 1, p_3 = 0\}$ set by master:

x_1	x_2	y_1	y_2	z_1	z_2
$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$
0					

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0	0	0	0	0	X

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0	0	0	0	1	

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0	0	0	0	0	X
0	0	0	0	1	X
0	0	0	1	0	X

...

Lazy Clause Generation (LCG)

Descendant of CP and SAT:

- CP-style propagators
- SAT-style conflict analysis

Operates on 'atomic constraints' $[\![x \geq k]\!]$, $[\![x = k]\!]$.

Key attributes (for our purposes):

- Conflict analysis
  - Cuts to explain failure.
- Activity-driven search
  - Focus on hard-to-satisfy subproblems.
- Phase-saving
  - Save successful partial assignments we find.

Implicit subproblems, with LCG

With $\{p_1 = 1, p_2 = 1, p_3 = 0\}$:

x_1	x_2	y_1	y_2	z_1	z_2
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0	0	0	0	0	X	$[\![p_3 \geq 1]\!] \vee [\![z_1 \geq 1]\!]$

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$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	1	X	$[\![p_3 \geq 1]\!]$

Most of the benefits of explicit partitioning, plus:

- Disjointness isn't required
- We get cuts for free

Strengthening cuts

The nogoods we obtain are usually not minimal.

- Choose a strict subset of the current cut, solve again.
  - If $\texttt{UNSAT}(C)$, we have a new, stronger cut.
  - If $\texttt{SAT}(\mu)$, at least one element is needed.
- Repeat until we find a minimal cut (or expend computation budget)

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However! The ‘subproblem’ is complete.
Thus μ is a feasible (though not optimal) solution.

We can then tighten bounds on the objective:

- in the master, to get earlier fathoming
- in the subproblem, to derive tighter cuts

A Dual viewpoint of Logic Based Benders

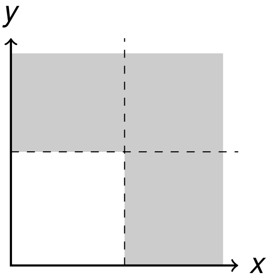
- Master (usual) perspective
  - Master solves relaxed problem
  - Subproblem solver extends master solution or adds cut
- Reversed perspective
  - Master generates a partial solution likely to be “good”
  - CP solver uses this as a basis for Large Neighbourhood Search to find good solutions

Representing cuts

Nogoods from the LCG solver are disjunctions of bounds.

$$[\![x \geq 10]\!] \vee [\![y \geq 10]\!]$$

Problem: Can't be directly expressed as a linear inequality.



Reifying bounds

Lazily introduce 0–1 variables for relevant bounds:

$$x \geq 0 + 10b_{[\![x \geq 10]\!]} + 5b_{[\![x \geq 15]\!]}$$
$$x < 10 + 5b_{[\![x \geq 10]\!]} + 35b_{[\![x \geq 15]\!]}$$

$$b_{[\![x \geq 10]\!]} \geq b_{[\![x \geq 15]\!]}$$

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Which we then use to express cuts:

$$b_{[\![x \geq 10]\!]} + b_{[\![y \geq 10]\!]} \geq 1$$

Outline

Experiments

Several classes of instances:

- Planning and scheduling
  Common LBBD benchmark
- Single-source capacitated plant location
  Pure MIP
- Job shop scheduling w. machine & order-dependent setup times
  TSP subproblem

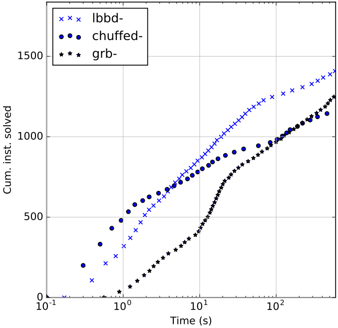
Comparing:

`chuffed` an LCG solver

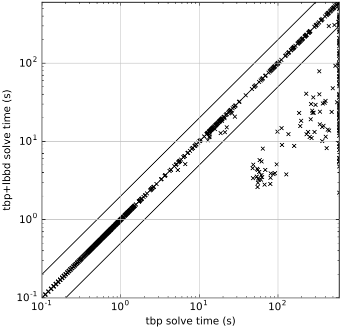
`Gurobi` a MIP solver

`mzn-lbbd` automatic LBBD method (using `Gurobi` and `chuffed`)

Results



instances solved



Theoretical best portfolio, with and without `mzn-lbbd`.

Results: Observations

- Doesn't strictly dominate either CP or MIP
  - but robust, and performs better in aggregate
- Not just best-of-both-worlds
  - Solves 79 instances not solved by either CP or MIP.
- Doesn't compete with Benders' methods with specialized (non-CP) subproblem solvers.
  - TSP subproblems, etc.

Outline

Conclusion

- Automatic Logic Based Benders provides a hybrid of
  - Integer Programming, and
  - Constraint Programming
- Takes advantage of the strengths of both methods
- One PhD worth of implementation is reduced to writing one model!

Further work

Many parameters to tune (globally, or per domain):

- Cut minimization strategy
- Generating multiple cuts
- Resource limits

Master currently includes no relaxation of omitted constraints.