Convex Feasibility via Monotropic Programming

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Outline

1. Problem Formulation
2. Monotropic Programming
3. Preliminaries
4. Facts
5. Analysis of Consistency

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The problem formulation

Let $H$ be a Hilbert space and let $C_n$, $n = 1, \ldots, m$ be convex closed subsets of $H$. The convex feasibility problem is to find some point

$$x \in \bigcap_{n=1}^{m} C_n \quad (CFP)$$

when this intersection is non-empty.
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The *CFP* has wide ranging applications:

- medical imaging, computerised tomography, signal processing.
- Partial differential equations (Dirichlet problem), complex analysis (Bergman kernels, conformal mappings);
- Subgradient algorithms with application in solution of convex inequalities, minimization of convex nonsmooth functions.
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Fact (Bauschke-Borwein, 1996)

- CFP equivalent to problem involving only two convex and closed sets in $H^m = H \times \ldots \times H$ consisting of $m$ copies of $H$, with the additional advantage that one of these sets is a linear subspace.
- Hence, from now on we assume that we are dealing with only two (possibly disjoint) closed convex sets.
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\[
\begin{align*}
\min & \sum_{i=1}^{m} f_i(x_i) \\
\text{subject to } & (x_1, \ldots, x_m) \in S,
\end{align*}
\]

- \( f_i : H_i \to \mathbb{R} \cup \{+\infty\} \) proper, convex,
- \( S \subseteq \prod_{i=1}^{m} H_i \) is a closed linear subspace

\((P)\) will be our primal model. 
\((P)\) has a very symmetric dual problem:
Monotropic Model (Minty, 1960)

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\text{max} & \quad \sum_{i=1}^{m} -f_i^*(x_i^*) \\
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- \( f_i^* : H_i \to \mathbb{R} \cup +\infty \) Fenchel conjugate of \( f_i \),
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1. Formulate CFP as a monotropic programming problem.
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Basic Ingredients:

- The Fenchel conjugate of $f$ is $f^* : H \to \mathbb{R} \cup \{+\infty\}$

  $$f^*(v) := \sup_{x \in H} \{ \langle v, x \rangle - f(x) \}$$

- The subdifferential of $f$ at $x$ is defined by

  $$\partial f(x) := \{ v \in H \mid \langle v, y - x \rangle \leq f(y) - f(x), \; \text{for all} \; y \in H \}$$

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Basic Ingredients (II):

- For $C \subset H$, the *indicator function* of $C$ is $\iota_C(x) := 0$ if $x \in C$ and $\iota_C(x) := +\infty$ otherwise.

- The *support function* of $C$ is
  \[
  \sigma_C(v) := \sup_{y \in C} \langle v, y \rangle
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  for $v \in H$

Easy to check \((\iota_C)^* = \sigma_C\)
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For \( \psi_1, \psi_2 : H \to \mathbb{R} \cup \{+\infty\} \), their *infimal convolution* is defined by

\[
(\psi_1 \Box \psi_2)(z) := \inf_{z_1 + z_2 = z} \{ \psi_1(z_1) + \psi_2(z_2) \}.
\]

For \( f : H \to \mathbb{R} \cup \{+\infty\} \) recall that the *epigraph* is the set

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epi f := \{(x, r) \in H \times \mathbb{R} : f(x) \leq r\}.
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Fact (B.-Jeyakumar, 2005):

$C, D \subset H$ closed convex:

$C \cap D \neq \emptyset \iff (0, -1) \notin \text{cl}(\text{epi} \sigma_C + \text{epi} \sigma_D)$
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Primal for CFP:

Our problem is (recall we reduced the problem to 2 sets):

\[
\text{find } (x, y) \in C_1 \times C_2 \subset H \times H, \text{ such that } x = y
\]

which can be formulated as

\[
\min_{(x, y) \in S} d_{C_1}(x) + d_{C_2}(y) \tag{P}
\]

where \( S = \{(x, y) \in H^2 : x = y\} \).
Using monotropic formulation we obtain its dual:

\[
\sup_{(v, w) \in S^\perp} - d^*_C(v) - d^*_C(w)
\]

where \( S^\perp = \{(u, v) \in H^2 : u + v = 0\} \).

What do we know about this primal-dual pair?
Dual for CFP:

Using monotropic formulation we obtain its dual:

\[
\sup_{(v, w) \in S^\perp} \left( d_{C_1}^*(v) - d_{C_2}^*(w) \right) \quad (D)
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What do we know about this primal-dual pair?
Duality facts:

Pro 15.22 and Theo 19.1 from Bauschke-Combettes book yield:

$$\nu(P) = \nu(D)$$ and (D) always has a solution

In this situation, \((x, y)\) solves \((P)\) and \((u, v)\) solves \((D)\). Moreover,

\[
\begin{align*}
(x, y) &\in S, \\
(u, v) &\in S^\perp, \\
u &\in \partial d_{C_1}(x), \\
v &\in \partial d_{C_2}(y)
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Proof not very direct!
Duality facts:

Pro 15.22 and Theo 19.1 from Bauschke-Combettes book yield:

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\[ (x, y) \in S, \quad (u, v) \in S_{\perp} \]
\[ u \in \partial d_{C_1}(x) \quad \quad v \in \partial d_{C_2}(y) \]

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\[ d^*_C(v) = \sigma_C(v) + \iota_B(v) \] yields:

\[
\sup_{v \in H} \left[ -d^*_C(v) - d^*_C(-v) \right] = \min_{t \in [0,1]} \left( \inf_{\|v\| \leq 1} \sigma_C(v) + \sigma_C(-v) \right),
\]

which gives an equivalent reformulation of the dual in terms of \( \Phi(1) \). Always \( \Phi(1) \leq 0 \). Value \( \Phi(1) \) gives important information:
\[ d_C^*(v) = \sigma_C(v) + \nu_B(v) \]
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\sup_{v \in H} - d_{C_1}^*(v) - d_{C_2}^*(-v) = \\
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Consistency results for CFP:

1. \( \Phi(1) < 0 \iff 0 \notin \text{cl}(C_2 - C_1) \). So \( C_1 \cap C_2 = \emptyset \).

2. \( \Phi(1) = 0 \iff 0 \in \text{cl}(C_2 - C_1) \). This leads to two cases:

   2.1 If \( (\sigma_{C_1} \square \sigma_{C_2}) \) is lsc at 0, then \( C_1 \cap C_2 \neq \emptyset \).

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Consistency results for CFP:

1. \( \Phi(1) < 0 \iff 0 \notin \text{cl} \,(C_2 - C_1) \). So \( C_1 \cap C_2 = \emptyset \).

2. \( \Phi(1) = 0 \iff 0 \in \text{cl} \,(C_2 - C_1) \). This leads to two cases:
   
   2.1 If \( (\sigma_{C_1} \sqcap \sigma_{C_2}) \) is lsc at 0, then \( C_1 \cap C_2 \neq \emptyset \).
   
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Characterization of Consistency:

Assume that \((\sigma_{C_1} \boxtimes \sigma_{C_2})(0) > -\infty\). Then \((\sigma_{C_1} \boxtimes \sigma_{C_2})\) is proper, and TFSAE:

(i) \(C_1 \cap C_2 \neq \emptyset\),

(ii) \((\sigma_{C_1} \boxtimes \sigma_{C_2})\) is lsc at 0,

(iii) \(\{0\} \times \mathbb{R} \cap \text{epi} (\sigma_{C_1} \boxtimes \sigma_{C_2}) = \{0\} \times \mathbb{R}_{+}\)

Consequently, if \(\text{epi} \sigma_{C_1} + \text{epi} \sigma_{C_2}\) is closed, then \(C_1 \cap C_2 \neq \emptyset\).
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Characterization of Consistency:

Assume that $(\sigma C_1 \boxminus \sigma C_2)(0) > -\infty$. Then $(\sigma C_1 \boxminus \sigma C_2)$ is proper, and TFSAE:

(i) $C_1 \cap C_2 \neq \emptyset$,

(ii) $(\sigma C_1 \boxminus \sigma C_2)$ is lsc at 0,

(iii) $\{0\} \times \mathbb{R} \cap \text{epi} (\sigma C_1 \boxminus \sigma C_2) = \{0\} \times \mathbb{R}_+$

Consequently, if $\text{epi} \sigma C_1 + \text{epi} \sigma C_2$ is closed, then $C_1 \cap C_2 \neq \emptyset$. 
Characterization of Consistency:

Assume that \((\sigma_{C_1} \square \sigma_{C_2})(0) > -\infty\). Then \((\sigma_{C_1} \square \sigma_{C_2})\) is proper, and TFSAE:

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Consequently, if \(\text{epi} \sigma_{C_1} + \text{epi} \sigma_{C_2}\) is closed, then \(C_1 \cap C_2 \neq \emptyset\).
Consistency for CFP in the critical case $v(D) = 0$:

Recall that $(D)$ always has solutions. Assume $v(D) = 0$. Then:

(a) If $v = 0$ is unique solution of $(D) \iff C_1 \cap C_2 \neq \emptyset$.

(b) $(D)$ has multiple solutions if and only if $C_1 \cap C_2 = \emptyset$. In this situation, every nonzero dual solution induces a possibly improper separation of the sets.
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Inconsistency for CFP in critical case \( d(C_1, C_2) = 0 \).

TFSAE:

(i) \((P)\) has no solutions.

(ii) \(0 \in \text{cl}(C_1 - C_2) \setminus (C_1 - C_2)\).

(iii) \(\sigma_{C_1} \square \sigma_{C_2}\) is not lsc at 0.

(v) \(\{0\} \times \mathbb{R}_{--} \cap \text{epi} (\sigma_{C_1} \square \sigma_{C_2}) \neq \emptyset\).
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