Automatic Logic-based Benders Decomposition with MiniZinc

Toby O. Davies and Graeme Gange and Peter J. Stuckey

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Department of Computing and Information Systems,
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Outline

1. Logic Based Benders Decomposition
2. MiniZinc
3. Automating Logic Based Benders
4. Experiments
5. Conclusion
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Multi-resource Scheduling

\[ \text{minimize} \sum_{r \in R} \text{cost of schedule for } r \]

\text{s.t.} \quad \forall t \in T. \text{ task } t \text{ is scheduled on some } r

\forall r \in R. \text{ schedule for } r \text{ is feasible}

Frequently:

- Objective is a linear combination of 0–1 variables
- Feasibility constraint is something nastily combinatorial
  - Cumulative resource capacities, bin packing, \ldots
How do we solve it?

Integer Programming?
- Extremely good at optimizing linear terms
- Tends to choke on the feasibility constraints
- Capacity constraints produce large, weak linearizations

Constraint Programming?
- Specialized reasoning for many combinatorial constraints
- Much weaker bounding than MIP.
- Only tightens objective bounds when defining variables change.

Davies, Gange, Stuckey
Automatic LBBBD
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Decompose the problem into a (MIP) master problem over shared variables, and several independent (CP) subproblems.

**Master:** Assign tasks to machines

**Subproblem:** Schedule tasks on a single machine
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1. Find an optimal solution $\mu$ to the master  
2. Search for a feasible extension of $\mu$ to each subproblem  
   - If all subproblems are feasible, we have found an optimum.  
   - Otherwise, add a cut to the master and restart.

In theory, cuts are derived by solving the inference dual.  
In practice, some form of generate-and-test.
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\[ M \]

\[ S_1 \quad S_2 \quad S_3 \]
Logic-Based Benders Decomposition

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![Diagram showing master problem (M) and subproblems (S1, S2, S3)]
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\[ M \land c \]

\[ \mu' \]

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[Diagram showing the relationship between the master problem and the subproblems]
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Automating Logic-based Benders Decomposition

An effective strategy, but sees surprisingly little use.

- Specialized implementation per-problem.
- One implementation per PhD
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Limitations to be aware of:

- Frequently all-or-nothing (optimal solution or none)
- Subproblems must be fully independent (no coupling)
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What elements do we need for automating LBBD?

1. Automatic partitioning into master/subproblems
2. Systematic extraction of cuts from arbitrary subproblems
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MiniZinc: A solver-independent modelling language

- solver-independent
  - supported by CP, MIP, SAT, SMT, and local search solvers
- high-level
  - encode combinatorial substructures directly as global constraints
- defacto standard for CP modelling

Hands On Session
Learn MiniZinc: Wednesday 28th: 11:00 - 12:30
MiniZinc: How it works

**MiniZinc** High-level model specification translates to ...

```plaintext
constraint forall (m in machines) (  
  cumulative(  
    [starts[j] | j in jobs],  
    [duration[j,m] | j in jobs],  
    [resource[j,m]*bool2int(assign[j] = m) | j in jobs],  
    capacities[m]  
  )
);
```

**FlatZinc** Variable declarations and primitive constraints

```plaintext
% ...  
constraint cumulative(X_INTRODUCED_233,  
  X_INTRODUCED_235,X_INTRODUCED_234,15);  
constraint int_lin_le([1,-1],[X_INTRODUCED_27,  
  objective],-6);  
% ...  
```

Flattening uses a solver-specific library of transformations.
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Automating ‘decomposition’

A simple strategy: MIP master, **single** CP subproblem.
- Master contains all linear inequalities (and corresponding variables).
- Subproblem contains **everything** (as if solving directly with CP).

![Diagram showing the relationship between MIP model, CP model, MIP solver, CP solver, and constraints.]
Implicit subproblems

With classical CP, this is a terrible idea.

\[ P = \begin{align*}
   & p_1 + p_2 + p_3 \leq 2 \\
   \land & x_1 + x_2 \leq p_1 \\
   \land & y_1 + y_2 \leq p_2 \\
   \land & z_1 = z_2 + p_3 \land z_1 \neq z_2
\end{align*} \]

Assuming \{p_1 = 1, p_2 = 1, p_3 = 0\} set by master:

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Lazy Clause Generation (LCG)

Descendant of CP and SAT:
- CP-style propagators
- SAT-style conflict analysis

Operates on ‘atomic constraints’ \([x \geq k], [x = k]\).

Key attributes (for our purposes):
- Conflict analysis
  - Cuts to explain failure.
- Activity-driven search
  - Focus on hard-to-satisfy subproblems.
- Phase-saving
  - Save successful partial assignments we find.
Implicit subproblems, with LCG

With \( \{ p_1 = 1, p_2 = 1, p_3 = 0 \} \):

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\( [p_3 \geq 1] \lor [z_1 \geq 1] \)
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<td>X</td>
</tr>
</tbody>
</table>

\[ [p_3 \geq 1] \lor [z_1 \geq 1] \]
\[ [p_3 \geq 1] \]

Most of the benefits of explicit partitioning, plus:

- Disjointness isn’t required
- We get **cuts for free**
Strengthening cuts

The nogoods we obtain are usually not minimal.

- Choose a strict subset of the current cut, solve again.
  - If \textsc{UNSAT}(C), we have a new, stronger cut.
  - If \textsc{SAT}(\mu), at least one element is needed.

- Repeat until we find a minimal cut (or expend computation budget)
Strengthening cuts

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However! The ‘subproblem’ is complete.
Thus $\mu$ is a feasible (though not optimal) solution.

We can then tighten bounds on the objective:

- in the master, to get earlier fathoming
- in the subproblem, to derive tighter cuts
A Dual viewpoint of Logic Based Benders

- **Master (usual) perspective**
  - Master solves relaxed problem
  - Subproblem solver extends master solution or adds cut

- **Reversed perspective**
  - Master generates a partial solution likely to be “good”
  - CP solver uses this as a basis for Large Neighbourhood Search to find good solutions
Representing cuts

Nogoods from the LCG solver are disjunctions of bounds.

\[
[ x \geq 10 ] \lor [ y \geq 10 ]
\]

**Problem:** Can’t be directly expressed as a linear inequality.
Reifying bounds

Lazily introduce 0–1 variables for relevant bounds:

\[ x \geq 0 + 10b_{[x \geq 10]} + 5b_{[x \geq 15]} \]
\[ x < 10 + 5b_{[x \geq 10]} + 35b_{[x \geq 15]} \]
\[ b_{[x \geq 10]} \geq b_{[x \geq 15]} \]
Lazily introduce 0–1 variables for relevant bounds:

\[
\begin{align*}
    x & \geq 0 + 10b_{[x \geq 10]} + 5b_{[x \geq 15]} \\
    x & < 10 + 5b_{[x \geq 10]} + 35b_{[x \geq 15]} \\
    b_{[x \geq 10]} & \geq b_{[x \geq 15]}
\end{align*}
\]

Which we then use to express cuts:

\[
\begin{align*}
    b_{[x \geq 10]} + b_{[y \geq 10]} & \geq 1
\end{align*}
\]
Experiments

Several classes of instances:

- Planning and scheduling
  Common LBBD benchmark
- Single-source capacitated plant location
  Pure MIP
- Job shop scheduling w. machine & order-dependent setup times
  TSP subproblem

Comparing:

- **chuffed** an LCG solver
- **Gurobi** a MIP solver
- **mzn-lbbd** automatic LBBD method (using **Gurobi** and **chuffed**)
Results

# instances solved

Theoretical best portfolio, with and without \texttt{mzn-lbbd}.
Results: Observations

- Doesn’t strictly dominate either CP or MIP
  - but robust, and performs better in aggregate
- Not just best-of-both-worlds
  - Solves 79 instances not solved by either CP or MIP.
- Doesn’t compete with Benders’ methods with specialized (non-CP) subproblem solvers.
  - TSP subproblems, etc.
Conclusion

- Automatic Logic Based Benders provides a hybrid of
  - Integer Programming, and
  - Constraint Programming
- Takes advantage of the strengths of both methods
- One PhD worth of implementation is reduced to writing one model!
Further work

Many parameters to tune (globally, or per domain):
- Cut minimization strategy
- Generating multiple cuts
- Resource limits

Master currently includes no relaxation of omitted constraints.