RISK AND RELIABILITY IN OPTIMIZATION UNDER UNCERTAINTY

Terry Rockafellar
University of Washington, Seattle

AMSI Optimise
Melbourne, Australia
18 Jun 2018

Decisions in the Face of Uncertain Outcomes

$\Longrightarrow$ especially serious modeling challenges in optimization

appropriate formulation of **constraints** and **objectives**?

Asset management in finance:

a portfolio must be chosen with intelligent safeguarding
performance known at best from history and economic factors
the threat of losses cannot be eliminated

Logistics in operations research:

supplies must be stockpiled to meet demands
distribution and replacement expense should be kept at bay
demands and costs are random variables, but partly guesswork

Design problems in engineering:

a structure must be engineered to withstand various impacts
possible impacts can only be estimated
potential for failure must be kept low

Reliability of Structures in Engineering

Engineering areas that are affected:

civil, mechanical, electrical, manufacturing, mining, ...

Examples of structures:

buildings, bridges, tunnels, reservoirs/dams, vehicle frames, ship hulls, airplane wings, offshore platforms . . .

Tasks that are influenced: design, maintenance, retrofit

Sources of uncertainty

- loads that a structure may experience in the future
  earthquakes, floods, wind storms, impacts in crashes
- variability in materials utilized or encountered
  unknown aspects of composition, imprecision in manufacture
- environmental and economic circumstances
  climate effects, evolving demands in usage

Objectives and Constraints in Stochastic Optimization

Typical considerations:

- minimizing costs, maximizing performance
- keeping various good properties at adequate levels
- preventing bad properties from reaching dangerous levels

Helpful resources:

- statistical methodology in assessing uncertainty
- choosing good values of design variables
- organizing good routines of maintenance and inspection

Technical challenges:

- avoiding formulations that suffer from DISCONTINUITY
- promoting formulations that take advantage of CONVEXITY

$\longrightarrow$ **all of this must be integrated with good modeling!**

$\longrightarrow$ guidelines for that must be understood and incorporated!

A Stochastic Framework of Uncertainty

General "cost" expression in decision-making:

$c(x, v)$ with $x =$ **decision** vector, $v =$ **data** vector

$$x = (x_1, \ldots, x_n), \qquad v = (v_1, \ldots, v_m)$$

Stochastic uncertainty:

v is replaced by a **random variable** vector $V = (V_1, \ldots, V_m)$

then the "cost" becomes a random variable: $\underline{c}(x) = c(x, V)$

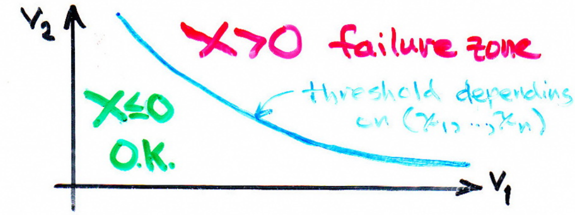
Key consequence:

the distribution of $\underline{c}(x)$ can only be **shaped** by the choice of x

but how then can constraints or minimization be understood?

Failure From a Standard Engineering Perspective

$X = c(x_1, \ldots, x_n; V_1, \cdots, V_r) =$ a "cost" that signals "danger"



Probability of failure: $p_f = \mathrm{prob}\left\{X > 0\right\}$

- How to compute or at least estimate?
- How to cope with dependence on $x_1, \ldots, x_n$ in optimization?

  both p_f and the threshold **shift** with changes in $x_1, \ldots, x_n$!

Troubles with this concept:

- poor mathematical behavior is a serious handicap
- failure probability ignores the **degree** of failure

Trade-offs and Preferences

Source of difficulties in modeling objectives and constraints

- nothing can be perfect, risks will almost always remain
- trade-offs must be contemplated between different risks
- preferences and intentions in handling risk must be articulated
- computational viability must be taken into account

Crucial issue for effectiveness:

How does the risk relate to the decision $(x_1, \ldots, x_n)$?
What are the mathematical properties of this dependence?

"**Coherent**" approaches to quantifying "risk" must play a role

A Broad Pattern for Quantifying Risk

Risk measures: functionals $\mathcal{R}$ that "**quantify the risk**" in a random variable X by a value $\mathcal{R}(X)$ ("risk" $\neq$ "uncertainty")

Systematic prescription

Faced with an uncertain "cost" $\underline{c}(x) = c(x, V)$ articulate it *numerically* as $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$ for a choice of risk measure $\mathcal{R}$

Constraints: keeping $\underline{c}(x)$ "sufficiently" $\leq b$

modeled as: constraint $\bar{c}(x) = \mathcal{R}(\underline{c}(x)) \leq b$

Objectives: keeping $\underline{c}(x)$ "sufficiently low"

modeled as: minimizing $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$

$=$ finding lowest b such that $\exists\, x$ with $\underline{c}(x)$ "$\mathcal{R}$-sufficiently $\leq$" b

Supporting theory: exploration of examples and guidelines

but note: choosing $\mathcal{R}$ expresses a decision-maker's preferences

Some Familiar Approaches Subject to Pros and Cons

[**random** cost $\underline{c}(x) = c(x, V)$ reduced to a **numerical** cost $\bar{c}(x)$]

Focusing on worst cases:

$\bar{c}(x) = \sup[\underline{c}(x)]$ [taking $\mathcal{R}(X) = \sup X$ (ess. sup)]

then $\bar{c}(x) \leq b \iff \underline{c}(x) \leq b$ almost surely

Passing to expectations:

$\bar{c}(x) = \mu[\underline{c}(x)] = E[\underline{c}(x)]$ [taking $\mathcal{R}(X) = \mu X = EX$]

then $\bar{c}(x) \leq b \iff \underline{c}(x) \leq b$ "on average"

Adopting a safety margin:

$\bar{c}(x) = \mu[\underline{c}(x)] + \lambda\sigma[\underline{c}(x)]$ [taking $\mathcal{R}(X) = \mu X + \lambda\sigma(X)$]

then $\bar{c}(x) \leq b$ unless in tail beyond λ standard deviations

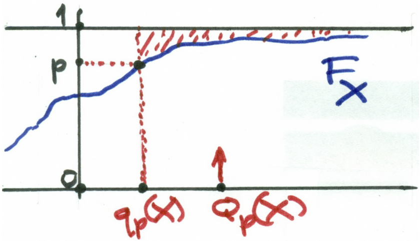
Looking at quantiles:

$\bar{c}(x) = q_p[\underline{c}(x)]$ [taking $\mathcal{R}(X) =$ quantile at level $p \in (0,1)$]

then $\bar{c}(x) \leq b \iff \mathrm{prob}\{\underline{c}(x) \leq b\} \geq p$

Quantiles and "Superquantiles": VaR and CVaR

F_X = cumulative distribution function for random variable X



Quantile: "value-at-risk" in finance

$$q_p(X) = \mathrm{VaR}_p(X) = F_X^{-1}(p)$$

Superquantile: "conditional value-at-risk" in finance

$$Q_p(X) = \mathrm{CVaR}_p(X) = E[X \mid X \geq q_p(X)] = \frac{1}{1-p}\int_p^1 q_t(X)dt$$

Replace quantiles by superquantiles in optimization?

take $\bar{c}(x) = Q_p(\underline{c}(x))$ in objective and constraint modeling?

then $\bar{c}(x) \leq b \iff \underline{c}(x) \leq b$ on average even in p-quantile-tail

Guidelines for Quantifying Risk in General

Key axioms for risk measures: $\mathcal{R} : \mathcal{L}^1(\Omega, \mathcal{A}, P) \to (-\infty, \infty]$

(R1) $\mathcal{R}(C) = C$ for all constants C

(R2) convexity, (R3) lower semicontinuity

Supplementary properties of interest:

(R4) positive homogeneity: $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for $\lambda > 0$

(R5) monotonicity: $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $X \leq X'$

(R6) aversity: $\mathcal{R}(X) > EX$ for nonconstant X

Coherent measure of risk: $\mathcal{R}$ satisfying (R1), (R2), (R3), (R5)

Averse measure of risk: $\mathcal{R}$ satisfying (R1), (R2), (R3), (R6)

Artzner et al. (2000) introduced coherency without aversity

Preservation of convexity under coherency of $\mathcal{R}$

$\underline{c}(x) = c(x, V)$ convex in x $\implies$ $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$ convex in x

a further advantage of coherency: it promotes **dualizations**

Coherency or its Absence

- $\mathcal{R}(X) = q_p(X) = \mathrm{VaR}_p(X)$ fails (R2), (R3), and (R6)!
- $\mathcal{R}(X) = Q_p(X) = \mathrm{CVaR}_p(X)$ satisfies **all** axioms
- $\mathcal{R}(X) = \sup X$ satisfies **all** axioms
- $\mathcal{R}(X) = \mu(X) = EX$ fails (R6) (coherency without aversity)
- $\mathcal{R}(X) = \mu(X) + \lambda\sigma(X)$, $\lambda > 0$, fails (R5) (no monotonicity)

$\Longrightarrow$ careful attention is needed in the choice of a risk measure!

Characterization of Coherent Risk Measures

Dual formula for $\mathcal{R}$

$$\mathcal{R}(X) = \sup_{Q \in \mathcal{Q}} \left\{ E[XQ] - \mathcal{J}(Q) \right\}$$

for some set $\mathcal{Q}$ of **probability densities** (i.e., $Q \geq 0$, $EQ = 1$) and some function $\mathcal{J}$ from $\mathcal{Q}$ to R such that $\inf_{Q \in \mathcal{Q}} \mathcal{J}(Q) = 0$

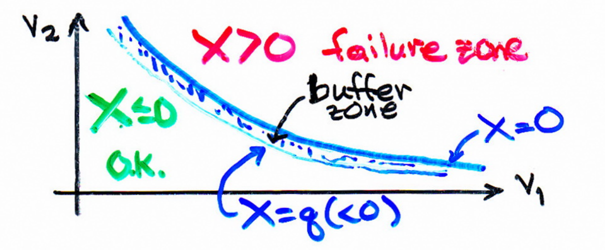
Interpretation: Q stands for a prob. measure P' contemplated as an **alternative** to the nominal prob. measure P (having $Q \equiv 1$)

- $\mathcal{J} \equiv 0$ on $\mathcal{Q}$ $\longleftrightarrow$ $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for $\lambda > 0$
- $\mathcal{J}(Q) = 0$ for $Q \equiv 1$ $\quad \Longrightarrow \quad$ $\mathcal{R}(X) \geq EX$

Buffered Failure — A Better Approach to Safety

Utilizing **superquantiles** in place of **quantiles** in reliability

the "danger cost": $X = c(x_1, \ldots, x_n; V_1, \cdots, V_r)$



Buffered probability of failure: $P_f = \mathrm{prob}\{X > q\}$

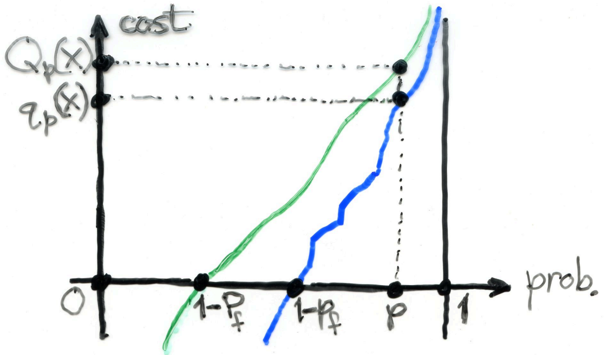
with q determined so as to make $E[\,X \mid X > q\,] = 0$!

Proposal: adjust failure modeling to P_f in place of p_f

safer by integrating tail information, and

easier also to work with in computation!

Failure Concept Relationships



$$q_p(X) = F_X^{-1}(p), \qquad Q_p(X) = \frac{1}{1-p}\int_p^1 q_s(X)\,ds$$

$q_p(X)$ can depend poorly on p, but $Q_p(X)$ depends nicely on p

failure modeling: p_f determined by $q_p(X) = 0,\ p = 1 - p_f$
P_f determined by $Q_p(X) = 0,\ p = 1 - P_f$

Some Practical Considerations

Key formula

$$Q_p(X) = \min_{C \in R} \left\{ C + \frac{1}{1-p} E\left[\max\{0, X - C\} \right] \right\} \quad p \in (0,1)$$

$$q_p(X) = \text{argmin} \quad \text{(if unique, otherwise the lowest)}$$

Application to failure: $X = \underline{c}\,(x) = c(x, V)$, want ≤ 0

ordinary probability of failure: constraint $p_f(\underline{c}\,(x)) \leq 1 - p$

corresponds having x satisfy $q_p(\underline{c}\,(x)) \leq 0$

buffered probability of failure: constraint $P_f(\underline{c}\,(x)) \leq 1 - p$

corresponds to having $Q_p(\underline{c}\,(x)) \leq 0$, hence equivalently to having x and C satisfy $C + \frac{1}{1-p} E\left[\max\{0, \underline{c}\,(x) - C\} \right] \leq 0$

Monte Carlo sampling: the expectation becomes a finite sum

research issue: what rules if the distribution of V depends on x?

Application to Objectives in Minimization

Recall notation: $\underline{c}(x) = c(x, V) =$ an uncertain "cost"
$=$ random variable depending on x and uncertain data V

Typical goal: keep the cost $\underline{c}(x) = c(x, V)$ as low as possible
(subject to constraints on the decision x)

Interpretation: minimize $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$ for a risk measure $\mathcal{R}$
i.e., minimize b such that $\exists x$ with $\underline{c}(x) - b$ $\mathcal{R}$-sufficiently ≤ 0

Risk-neutral approach: minimize $\bar{c}(x) = E[\underline{c}(x)]$
for $b = E[\underline{c}(x)]$, costs $\underline{c}(x) > b$ and $\underline{c}(x) < b$ average out

Risk-averse approaches: focus more on danger of cost **overruns**

Quantile version: minimize $q_p(\underline{c}(x))$ for some $p \in (0,1)$
min b with failure probability $p_f(\underline{c}(x) - b) < 1 - p$

Superquantile version: minimize $Q_p(\underline{c}(x))$ for some $p \in (0,1)$
min b with buffered failure probability $P_f(\underline{c}(x) - b) < 1 - p$

Some References

[1] R. T. Rockafellar, J. O. Royset (2010), "On buffered failure probability in design and optimization of structures," *Journal of Reliability Engineering and System Safety* [95], 499–510.

[2] R. T. Rockafellar, J. O. Royset (2015), "Engineering decisions under risk-averseness," *Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering* [1].

[3] R. T. Rockafellar, J. O. Uryasev (2018), "Minimizing buffered probability of exceedence by progressive hedging," *Mathematical Programming B*, submitted.

[4] R.T. Rockafellar, S.P. Uryasev (2013), "The fundamental risk quadrangle in risk management, optimization and statistical estimation," *Surveys in Operations Research and Management Science* [18], 33-53.

downloads: www.math.washington.edu/∼rtr/mypage.html