PROGRESSIVE HEDGING IN STOCHASTIC OPTIMIZATION

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Decisions and Observations

- Decisions can take advantage of information known in advance
- Information can come from observations and evolve in time
- Observations may come from realizations of random variables
- Optimization of decisions most cope with uncertainty and risk
- Evolution of information leads to problems of high dimension

Scenario model for information

Suppose that information comes in stages $k = 1, 2, \ldots, N$. Let $\xi_k \in \Xi_k$ stand for what is observed in stage $k$. Then $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ is a scenario.

Let $\Xi \subset \Xi_1 \times \cdots \times \Xi_N$ be the set of scenarios $\xi$ to be considered, and give it the structure of a probability space.

How might decisions and their optimization fit with this?
Uncertain Costs and Measures of Risk

Costs: not necessarily money, but low values preferred to high

Uncertainty: a cost can depend on the scenario: $X(\xi)$

$\Rightarrow$ then $X$ can be treated technically as a random variable

Risk measures: expressions $\mathcal{R}$ that assign to $X$ a risk level $\mathcal{R}(X)$

Examples of risk measures

- $\mathcal{R}(X) = E[X] = \mu(X)$, expected value or mean value
- $\mathcal{R}(X) = \mu(X) + \lambda \sigma(X)$ with $\lambda > 0$ giving risk margin
- $\mathcal{R}(X) = \sup X$, essential supremum: worst-case risk
- $\mathcal{R}(X) = \text{VaR}_\alpha(X)$, value-at-risk, $\alpha$-quantile
- $\mathcal{R}(X) = \text{CVaR}_\alpha(X)$, cond. value-at-risk, $\alpha$-superquantile

Coherency: axiomatized properties for $\mathcal{R}$ to be “good”

above, only #1, #3, and #5 are coherent
The Basics for a Single Stage of Information

Decisions: \( x \in \mathbb{R}^n \)  
Observations: \( \xi \in \Xi \)

Cost objective: \( g(x, \xi) \)  
Feasible set: \( C(\xi) \subseteq \mathbb{R}^n \)

Observation first? \( \implies \) the decision can depend on \( \xi \)  
get \( x(\xi) \) for each \( \xi \) by minimizing \( g(\cdot, \xi) \) over \( C(\xi) \)

Decision first? \( \implies \) the decision can’t depend on \( \xi \)  
choosing \( x \) just yields a random variable \( g(x, \cdot) : \xi \to g(x, \xi) \)  
minimize \( \mathcal{R}(g(x, \cdot)) \) in \( x \in \cap \xi C(\xi) \) for a risk measure \( \mathcal{R} \)

\( g(x, \xi) \) convex in \( x \), \( \mathcal{R} \) coherent \( \implies \mathcal{R}(g(x, \cdot)) \) convex in \( x \)

Possible mixture: the decision \( x \) could have two components  
- a first part which can’t depend on \( \xi \)  
- a second part which can depend on \( \xi \) \( \implies \) “recourse action”
Example: Two-Stage Stochastic Linear Programming

First stage decision: \( x_1 \in \mathbb{R}^{n_1} \)

Second stage decision: \( x_2(\xi) \in \mathbb{R}^{n_2} \)

Constraints: 
\[
\begin{align*}
    x_1 & \geq 0, \quad A_{11} x_1 \leq b_1 \\
    x_2 & \geq 0, \quad A_{22}(\xi) x_2(\xi) \leq b_1(\xi) - A_{21}(\xi) x_1
\end{align*}
\]

Objective: 
\[
\text{minimize } c_1 \cdot x_1 + E_{\xi} \left[ c_2(\xi) \cdot x_2(\xi) \right]
\]

Reduced problem:
\[
\text{minimize } c_1 \cdot x_1 + h(x_1) \text{ subject to } x_1 \geq 0, \quad A_{11} x_1 \leq b_1, \quad \text{where}
\]
\[
h(x_1) = E_{\xi} \left[ \min \left\{ c_2(\xi) \cdot x_2 \mid x_2 \geq 0, \quad A_{22}(\xi) x_2 \leq b_2(\xi) - A_{21}(\xi) x_1 \right\} \right]
\]

- the expectation in the objective signals risk neutrality
- replacing it by a risk measure \( \mathcal{R} \) can capture risk aversity
Interplay between decisions and observations in $N$ stages:

$x_1, \xi_1, x_2, \xi_2, \ldots, x_N, \xi_N$ with $x_k \in \mathbb{R}^{n_k}, \xi_k \in \Xi_k$

$x = (x_1, \ldots, x_N) \in \mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$

$\xi = (\xi_1, \ldots, \xi_N) \in \Xi \subset \Xi_1 \times \cdots \Xi_N$ scenarios

Nonanticipativity of decisions

$x_k$ can respond to $\xi_1, \ldots, \xi_{k-1}$ but not to $\xi_k, \ldots, \xi_N$:

$x(\xi) = (x_1, x_2(\xi_1), x_3(\xi_1, \xi_2), \ldots, x_N(\xi_1, \xi_2, \ldots, \xi_{N-1}))$

Simplifying assumptions for this talk:

- the scenario space $\Xi$ has only finitely many elements $\xi$
- each scenario $\xi \in \Xi$ has known probability $p(\xi) > 0$

$\rightarrow \Xi$ is a discrete probability space
Problem Formulation in this Setting — With Convexity

Problem ingredients: for each scenario $\xi \in \Xi$, let

\[ C(\xi) = \text{nonempty closed convex set in } \mathbb{R}^n \]

\[ g(x, \xi) = \text{continuous convex function of } x \in C(\xi) \]

Traditional stochastic programming — risk neutral

Over all nonanticipative decision choices

\[ x(\cdot) : \xi \rightarrow (x_1, x_2(\xi_1), x_3(\xi_1, \xi_2), \ldots, x_N(\xi_1, \xi_2, \ldots, \xi_{N-1})) \]

such that \( x(\xi) \in C(\xi) \) for all scenarios \( \xi \in \Xi \) minimize

\[ G(x(\cdot)) = E_\xi[g(x(\xi), \xi)] = \sum_{\xi \in \Xi} p(\xi)g(x(\xi), \xi) \]

Risk-averse objective as an alternative:

\[ G(x(\cdot)) = \text{CVaR}_\alpha(G(x(\cdot))) \text{ for the r.v. } G(x(\cdot)) : \xi \rightarrow g(x(\xi), \xi) \]

\[ \text{CVaR}_\alpha(X) = \text{conditional expectation in upper } \alpha \text{-tail of } X \]
Nonanticipativity Posed as a Subspace Constraint

Goal: introducing “Lagrange multiplier” for nonanticipativity

\[ \mathcal{L} = \text{all functions from scenario space } \Xi \text{ to decision space } R^n, \]
\[ x(\cdot) : \xi = (\xi_1, \ldots, \xi_N) \mapsto x(\xi) = (x_1(\xi), \ldots, x_N(\xi)) \]
\[ \Xi \subset \Xi_1 \times \cdots \Xi_N, \quad R^n = R^{n_1} \times \cdots \times R^{n_N} \]

Expectation inner product: for \( x(\cdot), w(\cdot) \in \mathcal{L} \)
\[ \langle x(\cdot), w(\cdot) \rangle = E_\xi [ x(\xi) \cdot w(\xi) ] = \sum_{\xi \in \Xi} p(\xi) \sum_{k=1}^N x_k(\xi) \cdot w_k(\xi) \]

Nonanticipativity subspace: with \( \xi = (\xi_1, \ldots, \xi_{k-1}, \xi_k, \ldots, \xi_N) \)
\[ \mathcal{N} = \{ x(\cdot) \in \mathcal{L} \mid x_k(\xi) \text{ depends only on } \xi_1, \ldots, \xi_{k-1} \} \]
\[ \quad \quad \rightarrow \quad x(\cdot) \text{ is nonanticipative} \iff x(\cdot) \in \mathcal{N} \]

Orthogonal complement:
\[ \mathcal{N}^\perp = \{ w(\cdot) \in \mathcal{L} \mid E_{\xi_k, \ldots, \xi_N} [ w_k(\xi_1, \ldots, \xi_{k-1}, \xi_k \ldots, \xi_N) ] = 0 \} \]
Projection Tool for Achieving Nonanticipativity

Let $\mathcal{P} = \text{projection onto } \mathcal{N}$, so that $\mathcal{I} - \mathcal{P} = \text{projection onto } \mathcal{N}^\perp$

$x(\cdot)$ nonanticipative $\iff [\mathcal{I} - \mathcal{P}](x(\cdot)) = 0$ (linear constraint)

Execution of projections relative to the information structure

- scenarios $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$ are information-equivalent at stage $k$ if $(\xi_1, \ldots, \xi_{k-1}) = (\xi'_1, \ldots, \xi'_{k-1})$
- let $A_k(\xi) = \text{kth-stage equivalence class containing } \xi$
- then $x(\cdot) = \mathcal{P}(\hat{x}(\cdot))$ has its $k$th-stage component given by

$$x_k(\xi) = \frac{\sum_{\xi' \in A_k(\xi)} p(\xi') \hat{x}_k(\xi')}{\sum_{\xi' \in A_k(\xi)} p(\xi')} \quad \text{(conditional expectation)}$$
minimize $G(x(\cdot))$ over all functions $x(\cdot) \in C \cap N \subset L$

$N = \text{the nonanticipativity subspace of } L$
$C = \{ x(\cdot) \in L \mid x(\xi) \in C(\xi) \subset \mathbb{R}^n \ \forall \xi \in \Xi \}$
$G(x(\cdot)) = E_{\xi}[g(x(\xi), \xi)] = \sum_{\xi \in \Xi} p(\xi) g(x(\xi), \xi)$

$C \subset L$ is closed convex, $G : C \rightarrow \mathbb{R}$ is continuous convex

Computational strategy:

• introduce **information cost** “multipliers” $w(\cdot) \in N^\perp$

• use these to relax the constraint of nonanticipativity

• thereby decompose the stochastic programming problem iteratively into solving “deterministic” subproblems
Progressive Hedging Approach (Rock. & Wets 1991)

- In iterations $\nu = 1, 2, \ldots$ solve “hindsight” problems for the separate scenarios $\xi$ in which the cost $g(x, \xi)$ is modified to
  
  $$g^\nu(x, \xi) + w^\nu(\xi) \cdot x + \frac{r}{2} \|x - x^\nu(\xi)\|^2 \quad (r = \text{parameter} > 0)$$

  with respect to the current $x^\nu(\cdot) \in N$ and $w^\nu(\cdot) \in N^\perp$

- This yields $\hat{x}^\nu(\xi)$ for each $\xi$, but the response function $\hat{x}^\nu(\cdot)$ won't be in $N$; restore nonanticipativity by projecting $\hat{x}(\cdot)$ onto $N$ and then generate an update for the information costs in $N^\perp$

- the risk-neutral case of $E_\xi[g^\nu(x(\xi), \xi)] = \sum_{\xi \in \Xi} p(\xi) g(x(\xi), \xi)$ supports this decomposition into separate scenario subproblems

- the risk-averse case with $\text{CVaR}_\alpha$ has no separability directly, but separability can be achieved, as will be explained later
Algorithm statement in the risk-neutral case with parameter $r > 0$

Having $x^\nu(\cdot) \in \mathcal{N}$ and $w^\nu(\cdot) \in \mathcal{M}$, get $\hat{x}^\nu(\cdot) \in \mathcal{L}$ by

$$\hat{x}^\nu(\xi) = \arg\min_{x \in C(\xi)} \left\{ g(x, \xi) + x \cdot w^\nu(\xi) + \frac{r}{2} \| x - x^\nu(\xi) \|^2 \right\}$$

Then get $x^{\nu+1}(\cdot) \in \mathcal{N}$ and $w^{\nu+1}(\cdot) \in \mathcal{N}^\perp$ by:

$$x^{\nu+1}(\cdot) = \mathcal{P}(\hat{x}^\nu(\cdot)), \quad w^{\nu+1}(\cdot) = w^\nu(\cdot) + r[\hat{x}^\nu(\cdot) - x^{\nu+1}(\cdot)]$$

Convergence theorem — when a solution pair $x(\cdot), w(\cdot)$, exists

The sequence $\{(x^\nu(\cdot), w^\nu(\cdot))\}_{\nu=1}^\infty$ generated by the algorithm will always converge to a particular solution pair $(x^*(\cdot), w^*(\cdot))$, with

$$\| x^{\nu+1}(\cdot) - x^*(\cdot) \|^2 + \frac{1}{r^2} \| w^{\nu+1}(\cdot) - w^*(\cdot) \|^2 < \| x^\nu(\cdot) - x^*(\cdot) \|^2 + \frac{1}{r^2} \| w^\nu(\cdot) - w^*(\cdot) \|^2$$
Adaptation of Progressive Hedging to Risk-Averse Case

**CVaR Minimization formula:** Rock. & Uryasev (2000, 2002)

\[
CVaR_\alpha(X) = \min_{z \in R} \left\{ z + \frac{1}{1-\alpha} E[\max\{0, X - z\}] \right\}
\]

**Consequence:** for the random variable \( G(x(\cdot)) : \xi \rightarrow g(x(\xi), \xi) \)

\[
CVaR_\alpha(G(x(\cdot))) = \min_{z \in R} \left\{ z + \frac{1}{1-\alpha} E_\xi[\max\{0, g(x(\xi), \xi) - z\}] \right\}
\]

**Risk-averse stochastic programming problem, expanded**

minimizing \( CVaR_\alpha(G(x(\cdot))) \) over \( x(\cdot) \in C \cup N \) is equivalent to

minimizing \( E_\xi[h(z, x(\xi), \xi)] \) over \( z \in R, x(\cdot) \in C \cup N \), where

\[
h(z, x(\xi), \xi) = z + \frac{1}{1-\alpha} \max\{0, g(x(\xi), \xi) - z\}
\]

- Incorporate \( z \) within \( x(\cdot) \) as an extra first-stage variable
- Then just apply the risk-neutral version of the progressive hedging algorithm with \( h \) taking the place of \( g \)
Some References


