On Numerical Methods for Spread Options

Joint work with Erik Schlögl

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The Valuation Problem I

- Fix the real-time horizon $\mathbb{T}$.
- Consider a filtered probability space

$$\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t<T<\mathbb{T}}\right)$$

- A Spread Option is an option written on the difference of two asset prices.
- i.e. its payoff is given by:

$$\mathcal{S}_T(S_1, S_2, K) = \max\{S_1(T) - S_2(T) - K, 0\}, \tag{1}$$

The Valuation Problem II

- Under a risk-neutral measure $\mathbb{Q}$, the value of Spread Option at time t is given by

$$\mathcal{S}_t(S_1, S_2, K) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_1(T) - S_2(T) - K)^+ \middle| \mathcal{F}_t\right]. \tag{2}$$

- Challenge: the numerairè change technique cannot be applied because of the existence of the non-zero strike ($K > 0$)
- If the log return for individual asset is log-normal distributed (as an example), the log return distribution of the difference is not known.
- This makes it very complicated to derive explicit formulae even in the well–celebrated framework of Black & Scholes (1973).
- One always resolves to numerical methods such as Monte Carlo and Fourier–based methodologies.

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Fourier–Based Approach

- Fast Fourier Transform (FFT) pricing methodology was coined by Carr & Madan (1999).
- Dempster & Hong (2002) were the first to derive FFT algorithms for correlation options and spread options.
- Spread options have a payoff with nonlinear exercise region.
- Dempster and Hong approximated the exercise region through a combination of rectangular strips.
- Then they applied FFT techniques on a regularized region to derive the upper and lower bounds for spread options value.
- Applying the methodologies of Dempster and Hong is computationally costly and not completely trivial.

Hurd & Zhou (2010) Approach

- Hurd & Zhou (2010) proposed an alternative and a most suitable version for FFT algorithms to pricing options in two and higher dimensions.
- Based on square integrable integral formulae for the payoff function.
- Assumes the knowledge of the model (joint) characteristic function.
- The price is given in terms of a complex gamma function.
- This method is proven to have

$$O(N^2 \log N)$$

complexity in comparison to $O(N^4)$ of a conventional Fourier transform and hence it is understood to be quite efficient.

Driving Processes

- Dempster & Hong (2002) applied:
  - 2-dimensional Geometric Brownian motion (GBM)
  - 3-Stochastic volatility processes (3-SV)
- In addition, Hurd & Zhou (2010) applied
  - Bilateral Variance Gamma processes (VG)

Our Approach

- Derive our results from the two-dimensional Parseval's type identity.
- Implement Hurd & Zhou (2010) methodology in a high–level programming language C++ and in an object–oriented approach.
- In addition to aforementioned processes, we applied 2-dimensional Normal Inverse Gaussian processes.
- Extend the pricing methodology to the fixed-income market.
- Provide insight of using FFTW library to compute FFT Backward– MATLAB *iff2* counterpart.

Fourier Transform

- The payoff function in Equation (1) can be re-written as

$$\mathcal{S}_T(S_1, S_2, K) = K \max\left\{\frac{S_1(T)}{K} - \frac{S_2(T)}{K} - 1, 0\right\} = K \max\{e^{x_1} - e^{x_2} - 1, 0\}. \tag{3}$$

- Let $X_{1t} = \log\left(\frac{S_1(T)}{K}\right)$ and $X_{2t} = \log\left(\frac{S_2(T)}{K}\right)$ with $X_{10} = x_1$ and $X_{20} = x_2$.
- Define

$$P(x_1, x_2) = \max\{e^{x_1} - e^{x_2} - 1, 0\}, \tag{4}$$

so that (3) becomes

$$\mathcal{S}_T(S_1, S_2, K) = KP(x_1, x_2). \tag{5}$$

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Spread Option Valuation

- Hurd & Zhou (2010) chose $\epsilon_1, \epsilon_2 \in \mathbb{R}$ such that $\epsilon_1 + \epsilon_2 < -1$ and $\epsilon_2 > 0$,
- then integrate along the lines L_1 and L_2 which runs from $\mathrm{i}\epsilon_1 - \infty \to \mathrm{i}\epsilon_1 + \infty$ and $\mathrm{i}\epsilon_2 - \infty \to \mathrm{i}\epsilon_2 + \infty$ respectively

$$
\begin{aligned}
\mathbb{V}_t(x_1, x_2) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{X}_0}\left[\left(e^{X_{1T}} - e^{X_{1T}} - 1\right)^+ \middle| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(s_1, s_2) q(s_1, s_2) ds_1 ds_2 \\
&= \frac{e^{-r(T-t)}}{4\pi^2} \int_{L_2}\int_{L_1} \hat{P}(s_1, s_2)\hat{q}(s_1, s_2) ds_1 ds_2 \\
&= \frac{e^{-r(T-t)}}{4\pi^2} \int_{L_2}\int_{L_1} e^{\mathrm{i}u_1 x_1 + \mathrm{i}x_2 u_2} \hat{P}(s_1, s_2)\Phi(s_1, s_2) ds_1 ds_2.
\end{aligned}
\tag{6}
$$

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2–Dimensional NIG

- Let X denote a $2d$ NIG random variable, i.e.

$$X = (X^1, X^2) = \mathsf{NIG}_2(\alpha, \mu, \delta, \beta, \Delta),$$

- where $\alpha, \delta \in \mathbb{R}_+$, $\mu, \beta \in \mathbb{R}^2$ and $\Delta \in \mathbb{R}^{2\times 2}$ symmetric, positive-definite matrix.

$$\Phi_X(u;T) = \exp\left(\langle \mathrm{i}u, \mu\rangle T + \delta T\left[\sqrt{\alpha^2 - \langle \beta, \Delta\beta\rangle} - \sqrt{\alpha^2 - \langle \beta + iu, \Delta(\beta + iu)\rangle}\right]\right)$$

Parameters

- Parameters taken from Hurd & Zhou (2010) paper

Table: Model inputs

S_1	S_2	r	T	d_1	d_2
100	96	0.1	1.0	0.05	0.05

- In all our results for FFT performance are benchmarked to both Monte Carlo with 1000000 simulation and 250 time steps, and we also provide the 95% level Monte Carlo confidence interval for Spread option value $\mathbb{V}$ given by:

$$\left(\mathbb{V} - z_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{N}}, \mathbb{V} + z_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{N}}\right),$$

where the quantity $\frac{s}{\sqrt{N}}$ is known as the *standard error*.

NIG

Table: NIG results: $\mu_1 = 0, \mu_2 = 0, \alpha = 6.20, \delta = 0.150$
$\beta_1 = -3.80, \beta_2 = -2.50, \rho = 0$

Strike K	FFT	Integration	Monte Carlo		
			value	95% conf. level	
2	9.49266	9.49119	9.48459	9.45828	9.51052
2.2	9.38858	9.38720	9.403	9.37675	9.42925
2.4	9.28489	9.28392	9.28919	9.26305	9.31533
2.6	9.18231	9.18136	9.18328	9.15721	9.20935
2.8	9.08077	9.07952	9.07361	9.0477	9.09952
3	8.98013	8.97840	8.99533	8.96951	9.02116
3.2	8.87969	8.87800	8.87543	8.84968	8.90118
3.4	8.77912	8.77834	8.77178	8.74624	8.79733
3.6	8.68022	8.67940	8.67697	8.65143	8.70251
3.8	8.58309	8.58119	8.55386	8.52848	8.57925

*Integration method by Caldana & Fusai (2013)

Monte Carlo bounds

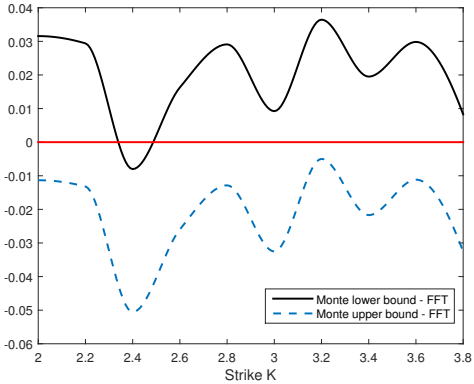

Figure: FFT prices deviation from the 95% level Monte Carlo confidence interval.

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Fixed-Income market

- We consider LIBOR formula developed in Alfeus, Grasselli and Schlögl (2017), i.e.

$$L(t,T) = \frac{1}{\delta}\left(\frac{\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T \phi(s)ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_c(s)+\lambda(s)q)ds}\right]} - 1\right). \tag{8}$$

- In particular, a case when LIBOR–OIS is driven solely by renewal risk
- The spread between LIBOR $L(T_{i-1},T_i)$ and $OIS(T_{i-1},T_i)$ with the contract margin K has the payoff

$$\delta\left(L(T_{i-1},T_i) - OIS(T_{i-1},T_i) - K\right)^+ \tag{9}$$

LIBOR-OIS Spread Options

The price at time $t \leq T_{i-1}$ is given by

$$\mathcal{S}_t = e^{-\int_t^{T_i} r_c(s)ds} \mathbb{E}_t^{\mathbb{Q}} \left[\delta \left(L(T_{i-1}, T_i) - OIS(T_{j-1}, T_j) - K \right)^+ \right] \tag{10}$$

$$= D^{\mathsf{OIS}}(t, T_i) \mathbb{E}_t^{\mathbb{Q}} \left[\delta \left(L(T_{i-1}, T_i) - OIS(T_{i-1}, T_i) - K \right)^+ \right] \tag{11}$$

$$= D^{OIS}(t, T_i) \mathbb{E}_t^{\mathbb{Q}} \left[\left(\frac{1}{\mathbb{E}_{T_{i-1}}^{\mathbb{Q}} \left[e^{-\int_{T_{i-1}}^{T_i} (r_c(s) + \lambda(s) q) ds} \right]} - \frac{1}{\mathbb{E}_{T_{i-1}}^{\mathbb{Q}} \left[e^{-\int_{T_{i-1}}^{T_i} (r_c(s) ds} \right]} - \delta K \right)^+ \right] \tag{12}$$

$$= D^{OIS}(t, T_i) \delta K \mathbb{E}_t^{\mathbb{Q}} \left[\left(\frac{1}{\delta K} \frac{1}{\mathbb{E}_{T_{i-1}}^{\mathbb{Q}} \left[e^{-\int_{T_{i-1}}^{T_i} (r_c(s) + \lambda(s) q) ds} \right]} - \frac{1}{\delta K} \frac{1}{\mathbb{E}_{T_{i-1}}^{\mathbb{Q}} \left[e^{-\int_{T_{i-1}}^{T_i} (r_c(s) ds} \right]} - 1 \right)^+ \right] \tag{13}$$

$$= D^{OIS}(t, T_i) \delta K \mathbb{E}_t^{\mathbb{Q}} \left[\left(e^{X_1(T_{i-1})} - e^{X_2(T_{i-1})} - 1 \right)^+ \right], \tag{14}$$

Cox et al. (1985) specification

Theorem

The conditional characteristic function of the random vector (X_1, X_2) is given by

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\mathrm{i}(u_1 X_1(T_{j-1})+u_2 X_2(T_{i-1}))}\right] &= e^{\mathrm{i}(u_1 f_1(T_{i-1})+u_2 f(T_{i-1})}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\mathrm{i}\sum_{n=1}^{d} y_n(T_{i-1})(u_1 g_1(n,T_{i-1})+u_2 g_2(n,T_{i-1}))}\right] \\ &= e^{\mathrm{i}(u_1 f_1(T_{i-1})+u_2 f(T_{i-1})+\sum_{n=1}^{d}\Phi_n(t,T_{i-1},0,-\mathrm{i}(u_1 g_1(n,T_{i-1}))+u_2 g_2(n,T_{i-1})))} \\ &\quad .e^{\sum_{n=1}^{d} y_n(t)\Psi_n(t,T_{i-1},0,-\mathrm{i}(u_1 g_1(n,T_{i-1})+u_2 g_2(n,T_{i-1}))}.\end{aligned}$$

The transform is well defined for all $u_1, u_2 \in \mathbb{C}$ and for all $n = 1, ..., d$.

$$\Im[u_1 g_1(n, T_{i-1}) + u_2 g_2(n, T_{i-1})] \geq -\frac{2\kappa_n}{\sigma_n^2} \tag{15}$$

where $\Im$ denotes the imaginary part of a complex number.

Numerical Example I

Table: CIR model calibrated parameters on 01/01/2013

	$y_i(0)$	κ_i	θ_i	σ_i	a_i	b_i
Factor	0.726349	0.1972365	0.1525022	0.187127	0.001036	0.051597
3	0.034559	0.9975629	0.01170301	0.008293	0.002325	0.044587
	0.09391	0.682302	0.2431752	0.407768	0.002373	0.010496
1	0.831418	0.2786581	0.7153432	0.22479	0.000517	0.000372

*$q = 0.6$ (see also Alfeus et al. (2017))

Numerical Example II

Table: LIBOR-OIS spread option prices, K taken to be the spot spread

				Monte Carlo		
T	**K**	**FFT**	**Integration**	**Low bound**	**Price**	**Upper bound**
0.5	0.0033825	0.0673846	0.070822817	0.071433	0.071437	0.07144
1	0.00166	0.0679472	0.071509611	0.0721315	0.0721317	0.0721319
2	0.0023	0.163379	0.163069141	0.165708	0.165709	0.165709
3	0.0034	0.1614773	0.161359976	0.163993	0.163994	0.163995
4	0.00507	0.1597959	0.15916084	0.161789	0.16179	0.161791
5	0.0071	0.1592471	0.156677915	0.159302	0.159302	0.159303
6	0.009278	0.153731	0.154014081	0.156631	0.156632	0.156633
8	0.012389	0.3979151	0.398282728	0.409938	0.40994	0.409943
9	0.012504	0.1476761	0.146416897	0.148962	0.148962	0.148963
10	0.01224	0.1452946	0.143098959	0.145583	0.145584	0.145584

Conclusion

- Full paper available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3099902
- Thank you for your attention!

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