On Numerical Methods for Spread Options

Joint work with Erik Schlögl

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The Valuation Problem I

- Fix the real-time horizon $\mathbb{T}$.
- Consider a filtered probability space

\[ (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t<T<\mathbb{T}}) \]

- A Spread Option is an option written on the difference of two asset prices.
- i.e. its payoff is given by:

\[ S_T(S_1, S_2, K) = \max\{S_1(T) - S_2(T) - K, 0\}, \quad (1) \]
The Valuation Problem II

• Under a risk-neutral measure $\mathbb{Q}$, the value of Spread Option at time $t$ is given by

$$S_t(S_1, S_2, K) = e^{-r(T-t)}\mathbb{E}^\mathbb{Q}\left[(S_1(T) - S_2(T) - K)^+|\mathcal{F}_t\right].$$

(2)

• Challenge: the numerairè change technique cannot be applied because of the existence of the non-zero strike ($K > 0$)

• If the log return for individual asset is log-normal distributed (as an example), the log return distribution of the difference is not known.

• This makes it very complicated to derive explicit formulae even in the well-celebrated framework of Black & Scholes (1973).

• One always resolves to numerical methods such as Monte Carlo and Fourier–based methodologies.
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Fourier–Based Approach

- Fast Fourier Transform (FFT) pricing methodology was coined by Carr & Madan (1999).
- Dempster & Hong (2002) were the first to derive FFT algorithms for correlation options and spread options.
- Spread options have a payoff with nonlinear exercise region.
- Dempster and Hong approximated the exercise region through a combination of rectangular strips.
- Then they applied FFT techniques on a regularized region to derive the upper and lower bounds for spread options value.
- Applying the methodologies of Dempster and Hong is computationally costly and not completely trivial.

- Hurd & Zhou (2010) proposed an alternative and a most suitable version for FFT algorithms to pricing options in two and higher dimensions.
- Based on square integrable integral formulae for the payoff function.
- Assumes the knowledge of the model (joint) characteristic function.
- The price is given in terms of a complex gamma function.
- This method is proven to have

\[ O(N^2 \log N) \]

complexity in comparison to \( O(N^4) \) of a conventional Fourier transform and hence it is understood to be quite efficient.
Driving Processes

- Dempster & Hong (2002) applied:
  - 2-dimensional Geometric Brownian motion (GBM)
  - 3-Stochastic volatility processes (3-SV)
- In addition, Hurd & Zhou (2010) applied
  - Bilateral Variance Gamma processes (VG)
Our Approach

- Derive our results from the two-dimensional Parseval’s type identity.
- In addition to aforementioned processes, we applied 2-dimensional Normal Inverse Gaussian processes.
- Extend the pricing methodology to the fixed-income market.
- Provide insight of using FFTW library to compute FFT Backward– MATLAB iff2 counterpart.
Fourier Transform

- The payoff function in Equation (1) can be re-written as

\[ S_T(S_1, S_2, K) = K \max \left\{ \frac{S_1(T)}{K} - \frac{S_2(T)}{K} - 1, 0 \right\} = K \max\{e^{x_1} - e^{x_2} - 1, 0\}. \]  

(3)

- Let \( X_{1t} = \log \left( \frac{S_1(T)}{K} \right) \) and \( X_{2t} = \log \left( \frac{S_2(T)}{K} \right) \) with \( X_{10} = x_1 \) and \( X_{20} = x_2 \).

- Define

\[ P(x_1, x_2) = \max\{e^{x_1} - e^{x_2} - 1, 0\}, \]  

(4)

so that (3) becomes

\[ S_T(S_1, S_2, K) = KP(x_1, x_2). \]  

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Spread Option Valuation

- Hurd & Zhou (2010) chose $\epsilon_1, \epsilon_2 \in \mathbb{R}$ such that $\epsilon_1 + \epsilon_2 < -1$ and $\epsilon_2 > 0$,
- then integrate along the lines $L_1$ and $L_2$ which runs from $i\epsilon_1 - \infty \to i\epsilon_1 + \infty$ and $i\epsilon_2 - \infty \to i\epsilon_2 + \infty$ respectively

$$
V_t(x_1, x_2) = e^{-r(T-t)}E_0^X \left[ (e^{X_1T} - e^{X_1T} - 1)^+ | \mathcal{F}_t \right]
$$

$$
= e^{-r(T-t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(s_1, s_2)q(s_1, s_2) ds_1 ds_2
$$

$$
= \frac{e^{-r(T-t)}}{4\pi^2} \int_{L_2} \int_{L_1} \hat{P}(s_1, s_2)\hat{q}(s_1, s_2) ds_1 ds_2
$$

$$
= \frac{e^{-r(T-t)}}{4\pi^2} \int_{L_2} \int_{L_1} e^{iu_1x_1 + iu_2x_2} \hat{P}(s_1, s_2)\Phi(s_1, s_2) ds_1 ds_2.
$$

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2–Dimensional NIG

• Let $X$ denote a 2d NIG random variable, i.e.

$$X = (X^1, X^2) = \text{NIG}_2(\alpha, \mu, \delta, \beta, \Delta),$$

• where $\alpha, \delta \in \mathbb{R}_+, \mu, \beta \in \mathbb{R}^2$ and $\Delta \in \mathbb{R}^{2 \times 2}$ symmetric, positive-definite matrix.

$$\Phi_X(u; T) = \exp \left( \langle iu, \mu \rangle T + \delta T \left[ \sqrt{\alpha^2 - \langle \beta, \Delta \beta \rangle} - \sqrt{\alpha^2 - \langle \beta + iu, \Delta (\beta + iu) \rangle} \right] \right)$$
Parameters

- Parameters taken from Hurd & Zhou (2010) paper

**Table: Model inputs**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>r</th>
<th>T</th>
<th>d1</th>
<th>d2</th>
</tr>
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<tbody>
<tr>
<td>S1</td>
<td>S2</td>
<td>0.1</td>
<td>1.0</td>
<td>0.05</td>
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</table>

- In all our results for FFT performance are benchmarked to both Monte Carlo with 1000000 simulation and 250 time steps, and we also provide the 95% level Monte Carlo confidence interval for Spread option value $V$ given by:

$$\left( V - z_{1-\frac{\alpha}{2}} \frac{s}{\sqrt{N}} , V + z_{1-\frac{\alpha}{2}} \frac{s}{\sqrt{N}} \right),$$

where the quantity $\frac{s}{\sqrt{N}}$ is known as the *standard error*. 
Table: NIG results: $\mu_1 = 0, \mu_2 = 0, \alpha = 6.20, \delta = 0.150$

$\beta_1 = -3.80, \beta_2 = -2.50, \rho = 0$

<table>
<thead>
<tr>
<th>Strike K</th>
<th>FFT</th>
<th>Integration</th>
<th>Monte Carlo</th>
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<tr>
<td></td>
<td>value</td>
<td>95% conf. level</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8.98013</td>
<td>8.97840</td>
<td>8.99533</td>
</tr>
<tr>
<td>3.2</td>
<td>8.87969</td>
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<td>3.4</td>
<td>8.77912</td>
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<td>3.6</td>
<td>8.68022</td>
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<td>3.8</td>
<td>8.58309</td>
<td>8.58119</td>
<td>8.55386</td>
</tr>
</tbody>
</table>

*Integration method by Caldana & Fusai (2013)
Figure: FFT prices deviation from the 95% level Monte Carlo confidence interval.
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Fixed-Income market

- We consider LIBOR formula developed in Alfeus, Grasselli and Schlögl (2017), i.e.

\[
L(t, T) = \frac{1}{\delta} \left( \frac{\mathbb{E}_t^Q \left[ e^{\int_t^T \phi(s) ds} \right]}{\mathbb{E}_t^Q \left[ e^{-\int_t^T (r_c(s) + \lambda(s) q) ds} \right]} - 1 \right).
\] (8)

- In particular, a case when LIBOR–OIS is driven solely by renewal risk
- The spread between LIBOR \( L(T_{i-1}, T_i) \) and \( OIS(T_{i-1}, T_i) \) with the contract margin \( K \) has the payoff

\[
\delta \left( L(T_{i-1}, T_i) - OIS(T_{i-1}, T_i) - K \right)^+.
\] (9)
LIBOR-OIS Spread Options

The price at time $t \leq T_i - 1$ is given by

$$S_t = e^{-\int_t^{T_i} r_c(s) \, ds} \mathbb{E}^Q_t \left[ \delta (L(T_i - 1, T_i) - OIS(T_{j-1}, T_j) - K)^+ \right]$$

(10)

$$= D^{OIS}(t, T_i) \mathbb{E}^Q_t \left[ \delta (L(T_i - 1, T_i) - OIS(T_{i-1}, T_i) - K)^+ \right]$$

(11)

$$= D^{OIS}(t, T_i) \mathbb{E}^Q_t \left[ \frac{1}{\mathbb{E}^Q_{T_i - 1}} \left[ e^{-\int_{T_i - 1}^{T_i} (r_c(s) + \lambda(s) q) \, ds} \right] - \frac{1}{\mathbb{E}^Q_{T_i - 1}} \left[ e^{-\int_{T_i - 1}^{T_i} (r_c(s) \, ds) - \delta K} \right] \right]^+$$

(12)

$$= D^{OIS}(t, T_i) \delta K \mathbb{E}^Q_t \left[ \frac{1}{\delta K} \left[ \frac{1}{\mathbb{E}^Q_{T_i - 1}} \left[ e^{-\int_{T_i - 1}^{T_i} (r_c(s) + \lambda(s) q) \, ds} \right] - \frac{1}{\mathbb{E}^Q_{T_i - 1}} \left[ e^{-\int_{T_i - 1}^{T_i} (r_c(s) \, ds) - 1} \right] \right] \right]^+$$

(13)

$$= D^{OIS}(t, T_i) \delta K \mathbb{E}^Q_t \left[ \left( e^{X_1(T_i - 1)} - e^{X_2(T_i - 1) - 1} \right)^+ \right]$$

(14)
Theorem

The conditional characteristic function of the random vector \((X_1, X_2)\) is given by

\[
\mathbb{E}_\mathcal{Q} \left[ e^{i(u_1 X_1(T_j-1) + u_2 X_2(T_i-1))} \right] = e^{i(u_1 f_1(T_i-1) + u_2 f(T_i-1))} \mathbb{E}_\mathcal{Q} \left[ e^{i \sum_{n=1}^{d} y_n(T_i-1) (u_1 g_1(n, T_i-1) + u_2 g_2(n, T_i-1))} \right]
\]

\[
= e^{i(u_1 f_1(T_i-1) + u_2 f(T_i-1) + \sum_{n=1}^{d} \Phi_n(t, T_i-1, 0, -i(u_1 g_1(n, T_i-1) + u_2 g_2(n, T_i-1)))}
\cdot e^{\sum_{n=1}^{d} y_n(t) \Psi_n(t, T_i-1, 0, -i(u_1 g_1(n, T_i-1) + u_2 g_2(n, T_i-1)))}.
\]

The transform is well defined for all \(u_1, u_2 \in \mathbb{C}\) and for all \(n = 1, \ldots, d\).

\[
\Im\left[u_1 g_1(n, T_i-1) + u_2 g_2(n, T_i-1)\right] \geq -\frac{2 \kappa_n}{\sigma_n^2} \quad (15)
\]

where \(\Im\) denotes the imaginary part of a complex number.
### Numerical Example I

**Table:** CIR model calibrated parameters on 01/01/2013

<table>
<thead>
<tr>
<th>Factor</th>
<th>$y_i(0)$</th>
<th>$\kappa_i$</th>
<th>$\theta_i$</th>
<th>$\sigma_i$</th>
<th>$\alpha_i$</th>
<th>$b_i$</th>
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<tbody>
<tr>
<td>3</td>
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<tr>
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<td>0.22479</td>
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<td>0.000372</td>
</tr>
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</table>

$q^*_q = 0.6$ (see also Alfeus et al. (2017))
Numerical Example II

Table: LIBOR-OIS spread option prices, $K$ taken to be the spot spread

<table>
<thead>
<tr>
<th>T</th>
<th>$K$</th>
<th>FFT</th>
<th>Integration</th>
<th>Low bound</th>
<th>Price</th>
<th>Upper bound</th>
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</thead>
<tbody>
<tr>
<td>0.5</td>
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<td>0.0721317</td>
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<tr>
<td>2</td>
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<td>0.163069141</td>
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<tr>
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<tr>
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<tr>
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<tr>
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Conclusion

- Thank you for your attention!
References


