About Stability of Error Bounds

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Outline

1. Local Error Bounds
   - Subdifferential Criteria
   - ε-perturbations. Radius of Error Bounds

2. Global Error Bounds
   - Subdifferential Criteria
   - ε-perturbations. Radius of Error Bounds

Error Bounds

Hoffman (1952); Łojasiewicz (1959);
Robinson (1975); Ioffe (1979); Mangasarian (1985); Auslender, Crouzeix (1988); Burke, Ferris (1993); Cornejo, Jourani, Zălinescu (1997); Pang (1997); Deng (1998); Klatte (1998); Lewis, Pang (1998); Ye (1998); Bauschke, Borwein, Li (1999); Studniarski, Ward (1999); Jourani (2000); Henrion, Outrata (2001, 2005); Ng, Zheng (2001); Azé, Corvellec (2002, 2004, 2014, 2017); Burke, Deng (2002, 2005); Henrion, Jourani (2002); Wu, Ye (2001, 2002, 2003); Azé (2003); Zălinescu (2003); Bosch, Jourani, Henrion (2004); Huang, Ng (2004); Ng, Yang (2004); Corvellec, Motreanu (2008); Ioffe, Outrata (2008); Ngai, Théra (2008, 2009); Penot (2010); Fabian, Henrion, Kruger, Outrata (2010, 2012); Ngai, Kruger, Théra (2010); Bednarczuk, Kruger (2012); Meng, Yang (2012); Chao, Cheng (2014); Kruger (2015, 2016)

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Local Error Bounds

X – Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $f(\bar{x}) = 0$,
$S_f := \{x \in X \mid f(x) \leq 0\}$

Definition

f has a local error bound at $\bar{x}$ with constant $\tau > 0$ if

$$\tau d(x, S_f) \leq f_+(x) \quad \text{for all } x \text{ near } \bar{x}$$

$f_+(x) := \max\{f(x), 0\}$

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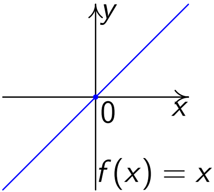

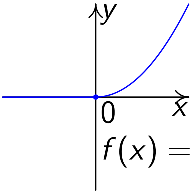

$$f(x) = \begin{cases} x^2, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

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- Convergence analysis
- Qualification conditions
- ...

Subdifferential Conditions

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Subdifferential Conditions

$$0 \notin \mathrm{bd}\,\partial f(\bar{x})$$

$$\Downarrow$$

$$\liminf_{x \to \bar{x},\ f(x) \downarrow 0} |\partial f|(x) > 0$$

$$\Updownarrow$$

$$\mathrm{Er}\, f(\bar{x}) > 0$$

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$\tau d(x, S_f) \leq f_+(x)$ for all x near $\bar{x}$

Example

$f(x) \equiv 0$

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Example

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x, & \text{if } x > 0 \end{cases}$$

ε-perturbations ($\varepsilon \geq 0$)

$$g \in \mathrm{Ptb}\,(f, \bar{x}, \varepsilon) \quad \Leftrightarrow \quad g(\bar{x}) = f(\bar{x}) \text{ and } \limsup_{x \to \bar{x}} \frac{|g(x) - f(x)|}{\|x - \bar{x}\|} \leq \varepsilon$$

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$g \in \mathrm{Ptb}\,(f, \bar{x}, \varepsilon)$ and $g - f$ is convex l.s.c.

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Linear ε-perturbations: $g \in \mathrm{Ptb}\,_l(f, \bar{x}, \varepsilon) \quad \Leftrightarrow$

$$g(x) - f(x) = \langle x^*, x - \bar{x} \rangle \quad (x \in X) \quad \text{and} \quad x^* \in \varepsilon \mathbb{B}^*$$

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$$\mathrm{Ptb}_l(f, \bar{x}, \varepsilon) \subset \mathrm{Ptb}_c(f, \bar{x}, \varepsilon) \subset \mathrm{Ptb}\,(f, \bar{x}, \varepsilon)$$

ε-perturbations and Radius of Error Bounds

$$\mathrm{Er}\,[\mathrm{Ptb}\,(f,\bar{x},\varepsilon)](\bar{x}) := \inf\{\mathrm{Er}\,g(\bar{x})|\ g \in \mathrm{Ptb}\,(f,\bar{x},\varepsilon),\ g \text{ is convex l.s.c.}\}$$

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$$\mathrm{Er}\,[\mathcal{P}_\varepsilon](\bar{x}) := \inf_{g\in\mathcal{P}_\varepsilon} \mathrm{Er}\,g(\bar{x})$$

$\mathcal{P}_\varepsilon = \mathrm{Ptb}\,_c(f,\bar{x},\varepsilon)$ or $\mathrm{Ptb}\,_l(f,\bar{x},\varepsilon)$

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$$\mathrm{Rad}\,[\mathcal{Q}](\bar{x}) := \inf\{\varepsilon > 0 \mid \mathrm{Er}\,[\mathcal{P}_\varepsilon](\bar{x}) = 0\}$$

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Radius of Error Bounds

X – Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – convex l.s.c., $f(\bar{x}) = 0$

Theorem

$$\mathrm{Rad}\,[\mathcal{Q}](\bar{x}) = |\partial f|_{\mathrm{bd}}(\bar{x})$$

$\mathcal{Q} = [\mathrm{general}]$ *or* $[\mathrm{convex}]$ *or* $[\mathrm{linear}]$ *perturbations*

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Corollary

$$\begin{aligned} 0 \le \varepsilon < |\partial f|_{\mathrm{bd}}(\bar{x}) \ &\Rightarrow\ \mathrm{Er}\,\{\mathrm{Ptb}\,(f, \bar{x}, \varepsilon)\}(\bar{x}) > 0 \\ &\Rightarrow\ \mathrm{Er}\,\{\mathrm{Ptb}\,_c(f, \bar{x}, \varepsilon)\}(\bar{x}) > 0 \\ &\Rightarrow\ \mathrm{Er}\,\{\mathrm{Ptb}\,_l(f, \bar{x}, \varepsilon)\}(\bar{x}) > 0 \ \Rightarrow\ \varepsilon \le |\partial f|_{\mathrm{bd}}(\bar{x}) \end{aligned}$$

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Global Error Bounds

X – Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – convex l.s.c.,
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$$|\partial f|_{\mathrm{bd}} := \inf_{f(x)=0} \{\|x^*\| \mid x^* \in \mathrm{bd}\, \partial f(x)\}$$

Subdifferential Criteria

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$$\widehat{|\partial f|}^{\varepsilon,\delta}_{\mathrm{bd}}(\bar{x}) = \begin{cases} \inf\limits_{f(x) \geq -\varepsilon\|x-\bar{x}\|-\delta} |\partial f|(x) & \text{if } 0 \notin \mathrm{int}\,\partial f(\bar{x}) \\ |\partial f|_{\mathrm{bd}}(\bar{x}) & \text{if } 0 \in \mathrm{int}\,\partial f(\bar{x}) \end{cases}$$

Subdifferential Criteria

X – Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – convex l.s.c., $f(\bar{x}) = 0$

$$|\partial f|(\bar{x}) = \inf\{\|x^*\| \mid x^* \in \partial f(\bar{x})\}$$
$$|\partial f|_{\rm bd}(\bar{x}) = \inf\{\|x^*\| \mid x^* \in {\rm bd}\,\partial f(\bar{x})\}$$

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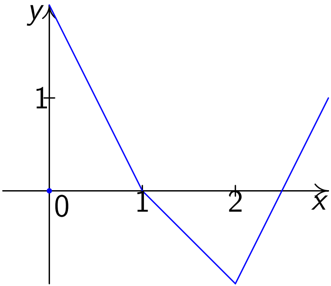
$$\widehat{|\partial f|}{}^{\varepsilon,\delta}_{\rm bd}(\bar{x}) \le |\partial f|_{\rm bd}, \quad \widehat{|\partial f|}{}^{0,0}_{\rm bd}(\bar{x}) = |\partial f|_{\rm bd}$$

Example

$$f(x) := \max\{-2x+2, -x+1, 2x-5\}, \quad x \in \mathbb{R}$$



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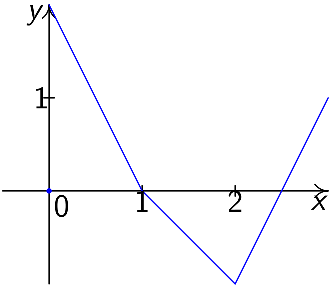

$$S_f = [1, 2.5],\ S_f^{=} = \{1, 2.5\}$$

$\widehat{|\partial f|}^{\varepsilon,\delta}_{\mathrm{bd}}(\bar{x})$

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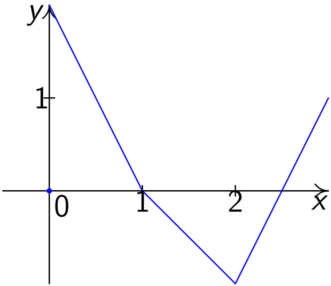

$S_f = [1, 2.5],\ S_f^{=} = \{1, 2.5\}$

$|\partial f|_{\mathrm{bd}} = 1, \quad \inf_{f(x)>0,\ x^* \in \partial f(x)} \|x^*\| = 2$

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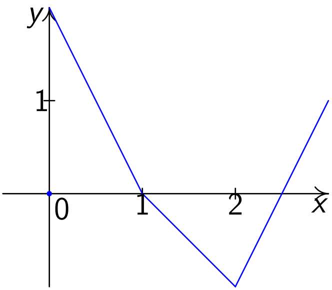

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$|\partial f|(2) = 0, \quad |\partial f(x)| \geq 1 \text{ if } x \neq 2$

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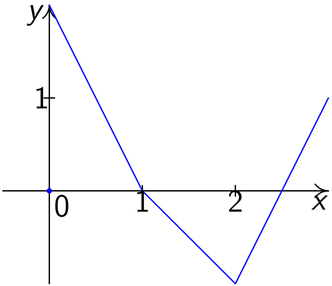

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$|\partial f|(2) = 0, \quad |\partial f(x)| \geq 1$ if $x \neq 2$

$\bar{x} = 1$:

$$\widehat{|\partial f|}^{\varepsilon,\delta}_{\rm bd}(1) = \begin{cases} 1 & \text{if } \varepsilon+\delta < 1 \\ 0 & \text{if } \varepsilon+\delta \geq 1 \end{cases}$$

ε-perturbations

$S_f \neq \emptyset$
$g \in \mathrm{Ptb}\,(f, \varepsilon) \;\Leftrightarrow S_g \neq \emptyset$, $g = f + p$, p convex,

ε-perturbations

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Linear ε-perturbations

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$$g(u) - f(u) = \langle x^*, u - x \rangle \quad (u \in X),$$

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$$\mathrm{Ptb}_l(f,\varepsilon) \subset \mathrm{Ptb}(f,\varepsilon), \quad \mathrm{Ptb}_l^w(f,\varepsilon) \subset \mathrm{Ptb}^w(f,\varepsilon)$$

ε-perturbations and Radius of Error Bounds

$$\mathrm{Er}\,[\mathcal{P}_\varepsilon] := \inf_{g \in \mathcal{P}_\varepsilon} \mathrm{Er}\, g$$

$\mathcal{P}_\varepsilon = \mathrm{Ptb}\,(f, \varepsilon)$ or $\mathrm{Ptb}\,_l(f, \varepsilon)$ or $\mathrm{Ptb}\,^w(f, \varepsilon)$ or $\mathrm{Ptb}\,_l^w(f, \varepsilon)$

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$\mathcal{Q}$: $\mathcal{Q}(f)$, $\mathcal{Q}_l(f)$, $\mathcal{Q}^w(f)$, $\mathcal{Q}^w_l(f)$

Radius of Error Bounds Estimates

X – Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ – convex l.s.c., $S_f \neq \emptyset$

Theorem

$$\mathrm{Rad}\,[\mathcal{Q}^w(f)] \leq \mathrm{Rad}\,[\mathcal{Q}^w_l(f)] \leq |\partial f|_{\mathrm{bd}} \leq \mathrm{Rad}\,[\mathcal{Q}(f)] \leq \mathrm{Rad}\,[\mathcal{Q}_l(f)]$$

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Corollary

$$\mathrm{Er}\,\{\mathrm{Ptb}\,^w(f,\varepsilon)\} > 0 \quad\Rightarrow\quad \mathrm{Er}\,\{\mathrm{Ptb}\,^w_l(f,\varepsilon)\} > 0 \quad\Rightarrow\quad \varepsilon \leq |\partial f|_{\mathrm{bd}}$$
$$\varepsilon < |\partial f|_{\mathrm{bd}} \quad\Rightarrow\quad \mathrm{Er}\,\{\mathrm{Ptb}\,(f,\varepsilon)\} > 0 \quad\Rightarrow\quad \mathrm{Er}\,\{\mathrm{Ptb}\,_l(f,\varepsilon)\} > 0$$

References

1. A. Kruger, H. Van Ngai, M. Théra, *Stability of error bounds for semi-infinite convex constraint systems*, SIAM J. Optimization **20** (2010), 2080–2096.
2. A. Kruger, H. Van Ngai, M. Théra, *Stability of error bounds for convex constraint systems in Banach spaces*, SIAM J. Optimization **20** (2010), 3280–3296.
3. A. Kruger, M. López, M. Théra, *Perturbation of error bounds*, Math. Program., Ser. B **168** (2018), 533–554.


Thank you