Potential flow of fluid from an elevated, two-dimensional source.

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Background

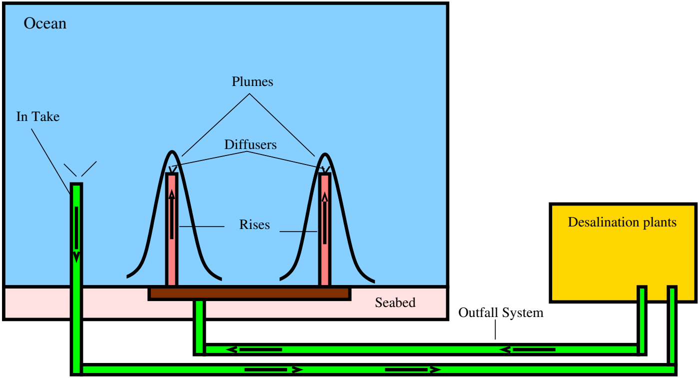
- Over 75 percent of the earth's surface is water.
- Saline water comprises 97.5 percent.
- Just 2.5 percent is potable water (drinkable).
- Desalination – removal of salt and other minerals from water.
- Get fresh water – human and animal consumption and agriculture.
- Desalination may be the key for new potable water.
- Plants exist throughout the World (2 in Western Australia). See (Matthew and Walter 2010) and(Tleimat 2012).

Some Types of Desalination

- Distillation
  - The water is converted to a steam – heating the ocean water.
  - This steam is collected, leaving the salt at the bottom.
  - Energy intensive – low pressure vessels.
- Reverse Osmosis
  - Two compartments, one saline, and one fresh.
  - Flow through a semi-permeable membrane – salt to fresh due to different pressure.
  - The water will flow in the opposite direction – filtering membrane.
  - Salt is filtered out.
  - The process operates with lower recoveries to save energy.

See (Lian-ying et.al. 2012) and (Tamim 2005)

Description of the Outfall System



Fresh water from the plant is used for drinking and irrigation while the brine component is returned to the ocean.

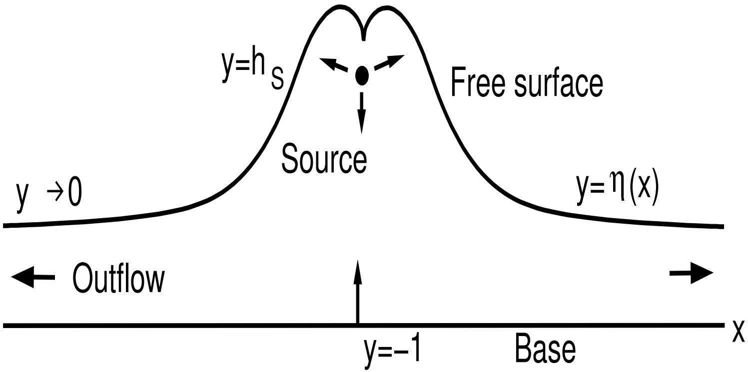
- The brine output component is returned to the ocean via a pipeline.
- The outfall – pipe along the seabed – the saline water offshore.
- Near the end of the pipeline – vertical risers –20 m high.
- These risers connect the outfall tunnel with diffusers
- The diffusers will distribute the brine into the seawater.

Ocean Plumes ....returning saline water to the ocean

- It is about 7 to 8 degrees warmer than ocean water, very salty.
- The discharged brine is always heavier than the ocean water.
- It will probably flow to the bottom as a plume of salty water.
- Possibility of killing fish and marine plants and damage reefs.
- Pump in higher up to decrease density of mixed water.
- Maximise mixing with ambient seawater.
- Need good models to Optimise Process.

Preliminary Model

- Fluid is pumped from an elevated source, flows downward, outward then hits the base.
- Seek steady, 2D flows from an elevated source above a flat base.
- Simplified (linearized solutions) are obtained at high flow rate.
- A numerical method is used to find fully nonlinear solutions and compared to linear.
- There is a minimum flow rate beneath which steady solutions do not exist.



Defining variables of the flow from an outfall plume.

Problem Formulation

- Assume 2D, irrotational flow of inviscid fluid.
- The velocity potential - $\phi(x, y)$ satisfies Laplace's equation $\nabla^2\phi = 0, y < \eta(x),$ where velocity $\mathbf{q} = (u, v) = \nabla\phi$
- No flow through interface $\phi_y = \phi_x \eta'(x)$ on $y = \eta(x)$,
- No flow through the base $\phi_y = \phi_x b'(x)$ on $y = b(x)$
- Source flow $\phi \to \frac{1}{\pi}\log(x^2 + (y - H_S)^2)^{1/2}$
- Pressure condition on the interface

$$\boxed{|\mathbf{q}|^2 + 2\mathbf{Fr}^{-2}\eta = \mathbf{1} \quad \text{on} \quad \mathbf{y} = \eta(\mathbf{x}).}$$

  - $\boxed{Fr = \sqrt{\frac{U^2}{gH}}.}$ =Froude number. Large Fr $\to$ high flow rate

Nekrasov Formulation

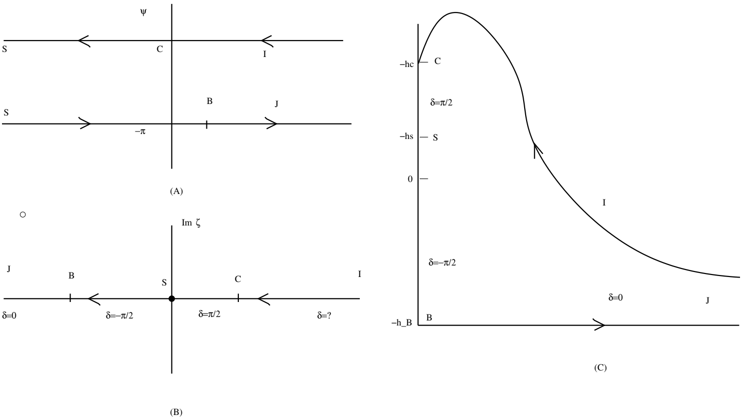
- Solve for potential function $\boxed{f = \phi + i\psi,}$ in complex plane mapping unknown surface to known boundary.

- Define $\boxed{f'(z) = \exp(-i\Omega), \quad \text{where} \quad \Omega = \delta + i\tau}$

  $\delta =$ angle of flow at any point and $\exp(\tau)$=velocity at any point.

- By using the transformation $\boxed{\exp(\pi f) = \zeta,}$ map to half plane

Nekrasov Formulation



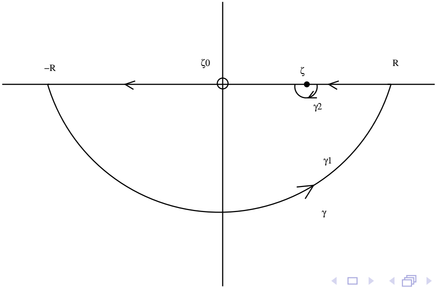
- (A) The complex velocity potential f-plane.
- (B) The lower half ζ-plane.
- (C) The real z-plane.

Nekrasov Formulation

- Applying Cauchy's integral formula in the ζ plane
- $$\Omega(\zeta) = -\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\Omega(\zeta_0)}{\zeta_0-\zeta}\,d\zeta_0.$$
- Taking real and imaginary parts gives δ and τ as

$$\tau(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\delta(\zeta_0)}{\zeta_0-\zeta}\,d\zeta_0,$$
$$\delta(\zeta) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau(\zeta_0)}{\zeta_0-\zeta}\,d\zeta_0,$$

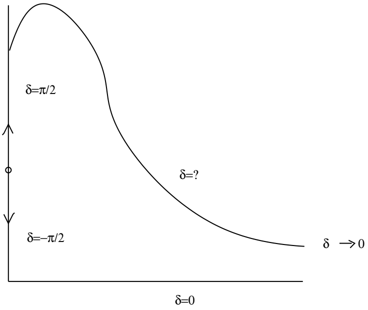
- The integrals are interpreted in the Cauchy principal value sense.



Nekrasov Formulation

- On the solid boundaries of the flow domain - the angle δ is known from the fact that the flow is along the boundary, so that

$$\delta = \begin{cases} 0, & \text{for} \quad -\infty < \zeta_0 < \zeta_B, \\ -\frac{\pi}{2}, & \text{for} \quad \zeta_B < \zeta_0 < 0, \\ \frac{\pi}{2}, & \text{for} \quad 0 < \zeta_0 < 1, \\ \text{Unknown}, & \text{for} \quad \zeta_0 > 1. \end{cases}$$



Nekrasov Formulation

- Substituting these values of $\delta(\zeta)$ into equation for $\tau(\zeta)$

$$\tau(\zeta) = \frac{1}{2}\log\left(\frac{(1-\zeta)(\zeta_B-\zeta)}{\zeta^2}\right) + \frac{1}{\pi}\int_1^\infty \frac{\delta(\zeta_0)}{\zeta_0-\zeta}d\zeta_0.$$

and letting $\delta = \arcsin\zeta^{-1/2} + \delta_B$ to simplify

$$\boxed{\tau(\zeta) = \tfrac{1}{2}\log\left(\tfrac{\zeta-\zeta_B}{\zeta}\right) + \tfrac{1}{\pi}\int_1^\infty \tfrac{\delta_B(\zeta_0)}{\zeta_0-\zeta}d\zeta_0, \quad 1<\zeta<\infty.}$$

$$\boxed{\exp(2\tau) + 2Fr^{-2}\int_1^\infty \exp(-\tau)\sin(\delta)\tfrac{\mathrm{d}\zeta}{\zeta} = 1, \quad 1<\zeta<\infty.}$$

- Solve for δ (eliminate τ), compute τ, then

$$x(\zeta) = \frac{1}{\pi}\int_1^\zeta \exp(-\tau)\cos(\delta)\frac{d\zeta_0}{\zeta_0},$$

$$y(\zeta) = \frac{1}{\pi}\int_\infty^\zeta \exp(-\tau)\sin(\delta)\frac{d\zeta_0}{\zeta_0}.$$

Linear Solution as $Fr \to \infty$ (high flow rate)

- As $Fr \to \infty$, $exp(2\tau) = 1$ on $y = \eta(x)$

- $$\boxed{\frac{1}{2}\log\left(\frac{\zeta-\zeta_B}{\zeta}\right) = -\frac{1}{\pi}\int_1^\infty \frac{\delta_B(\zeta_0)}{\zeta_0-\zeta}\mathrm{d}\zeta_0,\ 1 < \zeta < \infty}$$

- Using the transformation $\zeta = \cos^{-2}\frac{1}{2}\theta$, we obtain

$$\boxed{\frac{1}{2}\log\left(1-\zeta_B\cos^2\frac{\theta}{2}\right) = -\frac{1}{\pi}\int_0^\pi \delta_B(\theta_0)\left[\frac{\sin\theta_0}{\cos\theta-\cos\theta_0} + \frac{\sin\theta_0}{1+\cos\theta_0}\right]d\theta_0.}$$

- Can find exact solution as a Fourier series.

- Letting $\boxed{\delta_B(\theta) = \sum_{j=1}^{\infty} a_j \sin(j\theta),}$
- And noting that

  $\int_0^{\pi} \frac{\sin k\theta \sin\theta}{\cos\theta - \cos\theta_0} d\theta = -\pi \cos k\theta_0,$ (Tricomi, 1957) can find coefficients

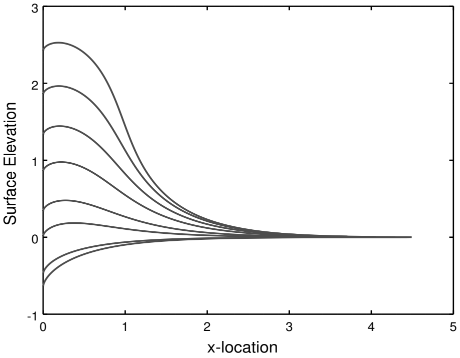
$$\boxed{a_j = -\frac{1}{\pi} \int_0^{\pi} \log\left|1 - \zeta_B \cos^2\frac{\theta}{2}\right| \cos j\theta d\theta, \quad j = 1, 2, ..., \infty.}$$

- Free surface point from

$$x(\theta_0) = \frac{1}{\pi} \int_{\pi}^{\theta_0} \cos\left(\frac{\pi}{2} - \frac{\theta}{2} + \sum_{j=1}^{\infty} a_j \sin(j\theta)\right) \tan\frac{\theta}{2} \mathrm{d}\theta,$$

$$y(\theta_0) = \frac{1}{\pi} \int_{0}^{\theta_0} \sin\left(\frac{\pi}{2} - \frac{\theta}{2} + \sum_{j=1}^{\infty} a_j \sin(j\theta)\right) \tan\frac{\theta}{2} d\theta.$$

- " Exact" solutions can be found for $\zeta_B = 0, -8, \infty.$ (to check)



- Free surface profile for $Fr \to \infty$ as source height increases.
- Each source is located just below the corresponding cusp point.
- Source near bottom, the free surface rises from the cusp and then levels off
- When source is higher, it rises to a maximum height and then falls downward before levelling off.

Table 1

Table 1		
a_k	*source height* $= 0.5$	*source height* $= 10$
a_1	1.7157e-01	9.2527e-01
a_2	-1.4719e-02	-4.5889e-01
a_3	1.6835e-03	3.0109e-01
a_4	-2.1664e-04	-2.2117e-01
a_5	2.9735e-05	1.7275e-01
a_{10}	-2.2105e-09	-7.5352e-02
a_{20}	-2.4412e-17	-2.8399e-02
a_{30}	2.8243e-19	-1.4269e-02
a_{40}	3.3494e-18	-8.0651e-03
a_{100}	-1.1480e-19	-5.9111e-04

- Convergence of series coefficients.
- Rapid for small source height.
- Slow for large source height.

Integral Equation for the Nonlinear Solution

$$\boxed{\tau(\zeta) = \frac{1}{2}\log\left(\frac{\zeta-\zeta_B}{\zeta}\right) + \frac{1}{\pi}\int_1^\infty \frac{\delta_B(\zeta_0)}{\zeta_0-\zeta}d\zeta_0, \quad 1 < \zeta < \infty.}$$

$$\boxed{\exp(2\tau) + 2Fr^{-2}\int_1^\infty \exp(-\tau)\sin(\delta)\frac{\mathrm{d}\zeta}{\zeta} = 1, \quad 1 < \zeta < \infty.}$$

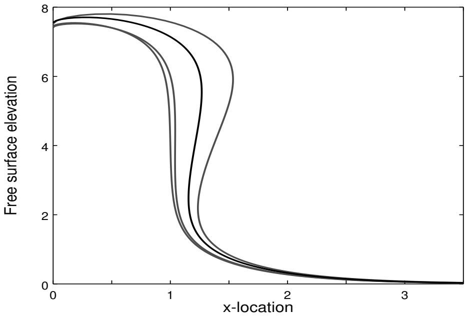
- $\zeta = \exp(\xi)$ for $1 < \zeta < \infty$ and so $0 < \xi < \infty$.
- Discretize equations $0 < \xi_k < \xi_N$, i.e. $\delta_k,\ k = 0, 1, 2, \ldots, N$ where ξ_N is some large value of ξ.
- Removing the singularity by noting that

$$\int_0^{\xi_N} \frac{\delta_B(\xi)}{1 - e^{\xi_0 - \xi}}d\xi = \int_0^{\xi_N} \frac{\delta_B(\xi) - \delta_B(\xi_0)}{1 - e^{\xi_0-\xi}}d\xi + \delta_B(\xi_0)\log\left(\frac{1 - e^{\xi_0 - \xi_N}}{1 - e^{\xi_0}}\right).$$

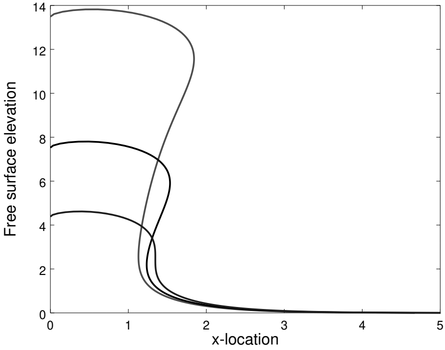
- Solve using MATLAB (Fsolve)

Results of Nonlinear Solution

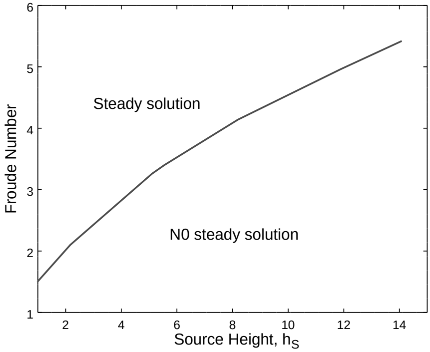
- Method converged to graphical accuracy with N=200 points on free surface..
- Ran in seconds on desktop.
- Solutions for large Fr matched linear solutions exactly.
- For each source height a minimum Fr exists.
- No steady solutions $Fr < Fr_{min}$.



- The shape of the plume at fixed source height $y_s \approx 7.3$ for differing Froude numbers.
- $Fr = 51, 10, 5$ and 4.14, moving left to right.
- Note that the difference between $Fr = 51$ and $Fr = 10$ is small.
- Bulbous top forms as Fr decreases.



- The shape of the plume at minimum Froude number for several different source heights.
- Tallest to shortest, these correspond to source elevations $y_S = 13.083, 7.195$ and 4.108 with corresponding $Fr = 30, 17$ and 10
- Taller plumes have overhanging portions may collapse.
- $Fr < Fr_{min}$ probably unsteady flows.



- The minimum values of Froude number at which steady solutions exist as a function of the source height when the number of coefficients $N = 240$.
- As the source height increases, the minimum value of Froude number with steady solutions increases.

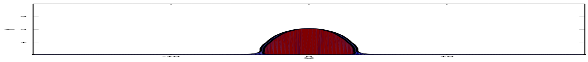
Concluding Remarks on Inviscid Solution

- Solutions have been obtained to the flow from an elevated two-dimensional source above a horizontal base.
- At high values of flow, the nonlinear solutions match the linear $Fr \to \infty$ solution.
- At each value of source height there is a minimum Froude number above which steady solutions exist.
- As the height of the source increases, this minimum value of Froude number, Fr, increases as a higher flow rate is required to maintain a steady flow.
- At higher elevations, minimum Fr solution surface has overhanging region.
- At lower elevations of the source, the overhang does not form and so the minimum is related to some other, as yet unknown, physical effect, but which may just be that the solutions become unsteady as suggested by the work of Turner(1966) and Stokes (2008).

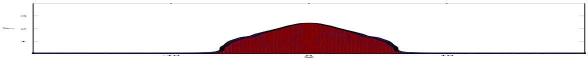
Optimal Situation

- A model developed that shows basic features of the flow.
- Should we design the system to have unsteady flow, to increase mixing?
- Should we have high flow rates?
- Should we oscillate the flow?
- Some preliminary results are shown over for source on the bottom.
- Will extend to include source off the bottom.

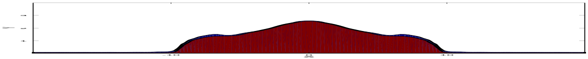
Simulation Using Navier-Stokes Equations



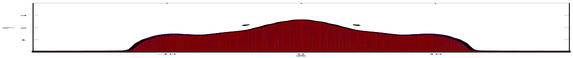
The contour for Fr=0.5,t=10



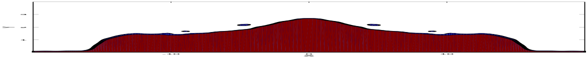
The contour for Fr=0.5,t=20



The contour for Fr=0.5,t=30



The contour for Fr=0.5,t=40



The contour for Fr=0.5,t=50

- A source situated on the bottom, circular outflow of dense fluid.
- Solution using vorticity formulation and Fourier methods.
- Takes several hours to run with 40 Fourier coefficients.
- The interface stays very sharp suggests previous (simpler model) quite good.
- Work on this model continuing.

Using these two models we can design an optimal outfall system.

Thanks For Listening