Bilevel Optimization and Pricing Problems

Martine Labbé

Computer Science Department
Université Libre de Bruxelles

INOCS Team, INRIA Lille

Follower

Leader

AMSI - Perth - June 2019

Outline

- Bilevel optimisation
- Linear bilevel optimization
- Pricing problem

PART I:
Bilevel optimization

Bilevel Optimization Problem

$$
\begin{array}{rl}
\max\limits_{x,y} & f(x,y) \\
\text{s.t.} & (x,y) \in X \\
 & y \in S(x) \\
\text{where} & S(x) = \operatorname*{argmax}\limits_{y} g(x,y) \\
 & \text{s.t.} (x,y) \in Y
\end{array}
$$

First paper on bilevel optimization

Bracken & McGill (OR,1973): First bilevel model, structural properties, military application.

Mathematical Programs with Optimization Problems in the Constraints

Jerome Bracken and James T. McGill

Institute for Defense Analyses, Arlington, Virginia

(Received October 5, 1971)

This paper considers a class of optimization problems characterized by constraints that themselves contain optimization problems. The problems in the constraints can be linear programs, nonlinear programs, or two-sided optimization problems, including certain types of games. The paper presents theory dealing primarily with properties of the relevant functions that result in convex programming problems, and discusses interpretations of this theory. It gives an application with linear programs in the constraints, and discusses computational methods for solving the problems.

Adequate framework for Stackelberg game

- Leader: 1st level,
- Follower: 2nd level,
- Leader takes follower's optimal reaction into account.

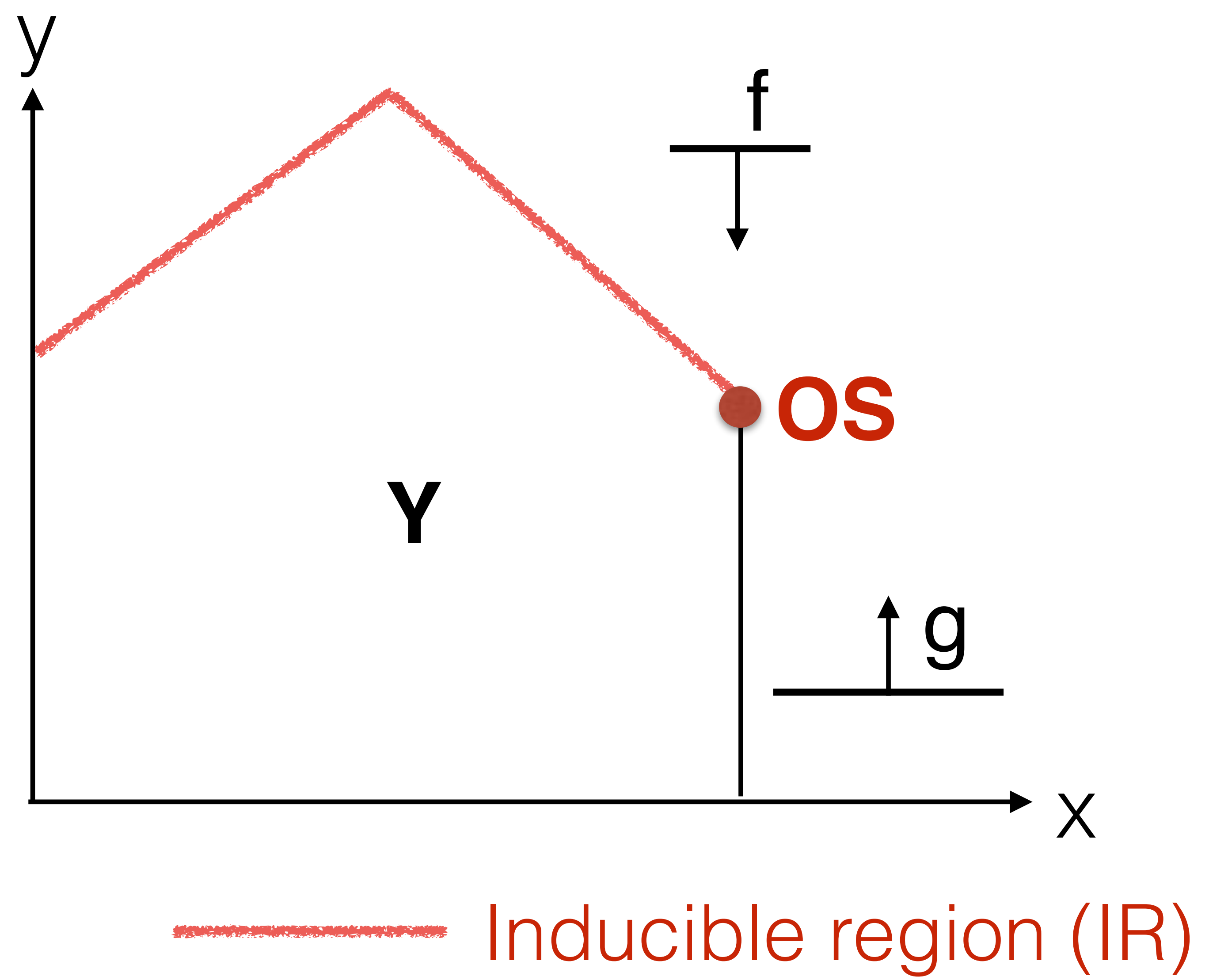
Heinrich von Stackelberg
(1905 - 1946)

Applications

- Economic game theory
- Production planning
- Revenue management
- Security
- …

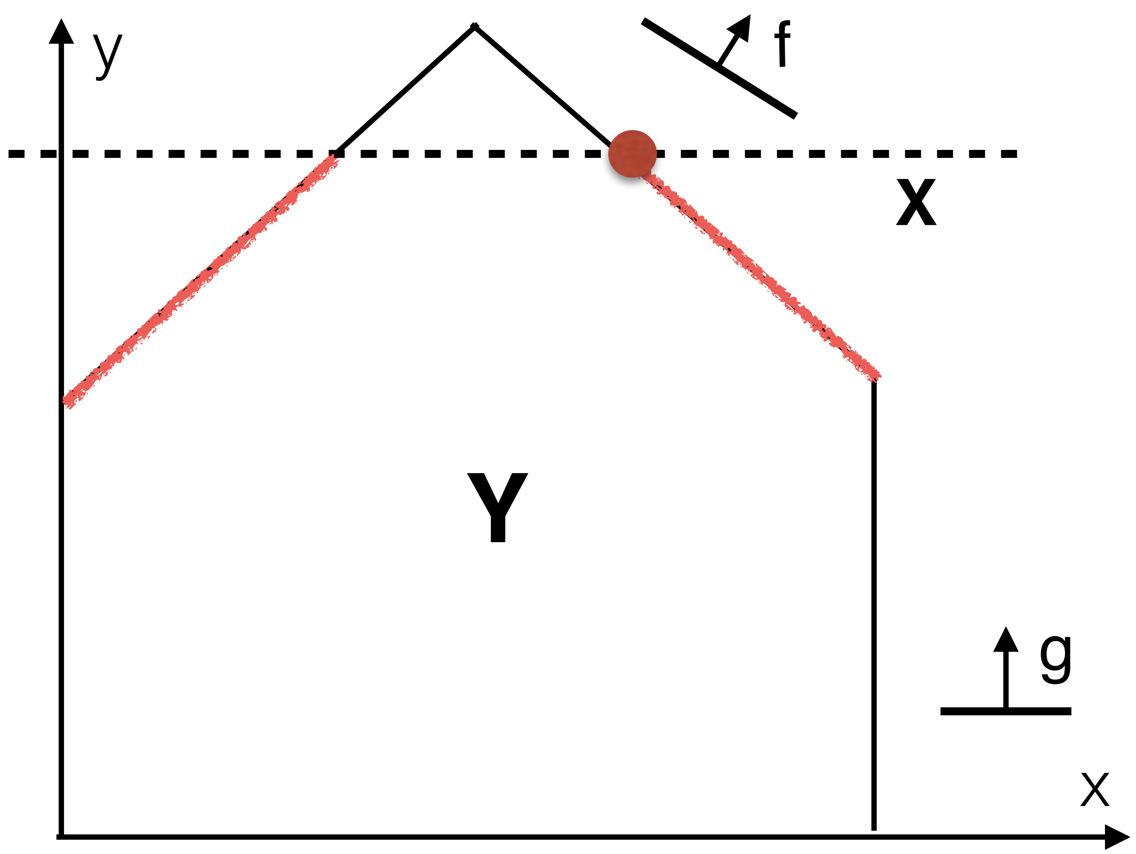
Example: a linear BP

$$
\begin{aligned}
\max_{x,y} \quad & f_1 x + f_2 y \\
\text{s.t.} \quad & \max_{y} g_1 x + g_2 y \\
& \text{s.t.} (x, y) \in Y
\end{aligned}
$$



Coupling constraints

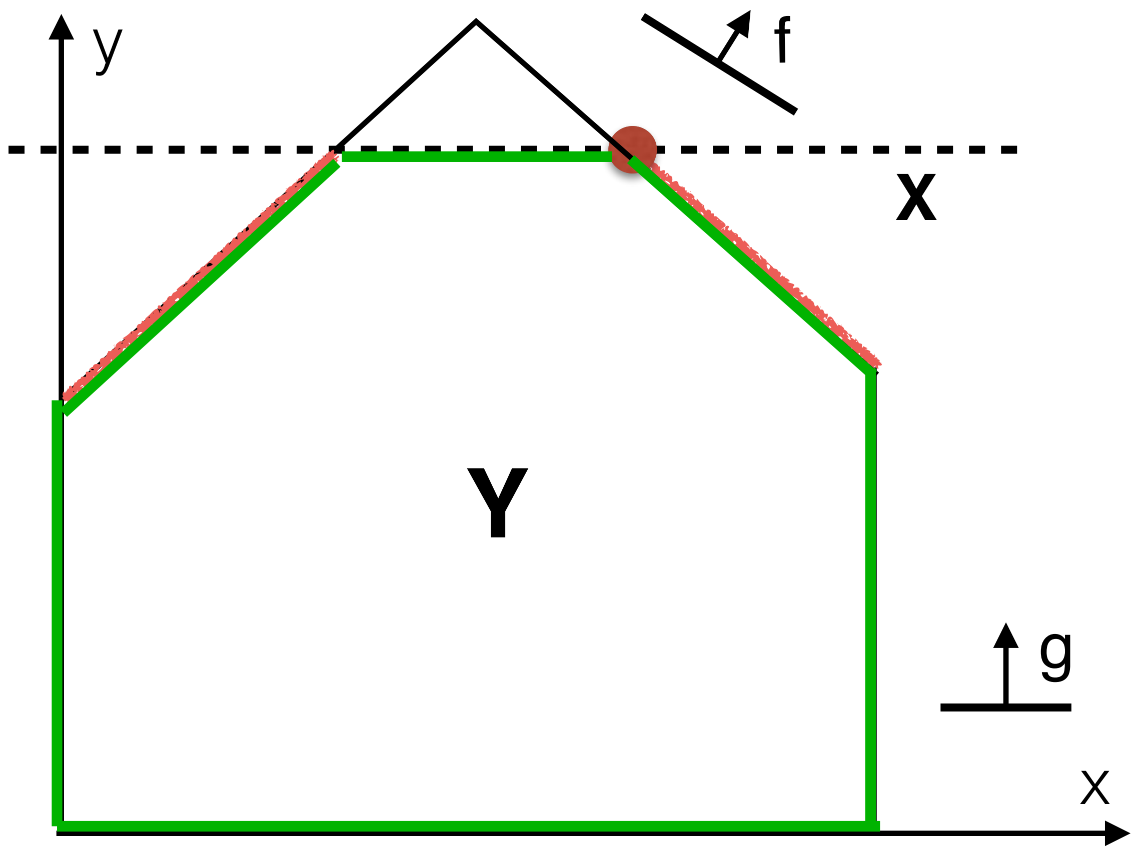
The follower sees only the second level constraints



$$\max_{x,y} \quad f_1 x + f_2 y$$
$$\text{s.t.} \quad (x, y) \in X$$
$$\max_{y} g_1 x + g_2 y$$
$$\text{s.t.} (x, y) \in Y$$

Coupling constraints

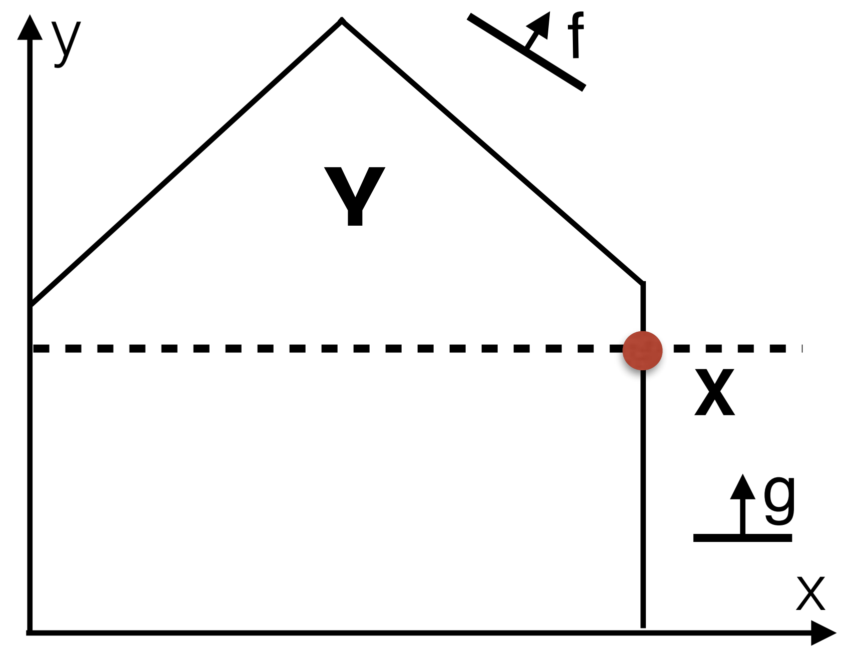
The follower sees only the second level constraints



$$\max_{x,y} \quad f_1 x + f_2 y$$
$$\text{s.t.} \quad (x,y) \in X$$
$$\max_{y} g_1 x + g_2 y$$
$$\text{s.t.} (x,y) \in Y$$

$$X \cap Y = \text{High Point Relaxation (HPR)}$$

Coupling constraints



Infeasible BP

$$\begin{array}{ll} \max\limits_{x,y} & f_1 x + f_2 y \\ \text{s.t.} & (x,y) \in X \\ & \max\limits_{y} g_1 x + g_2 y \\ & \text{s.t.} (x,y) \in Y \end{array}$$

Feasible BP

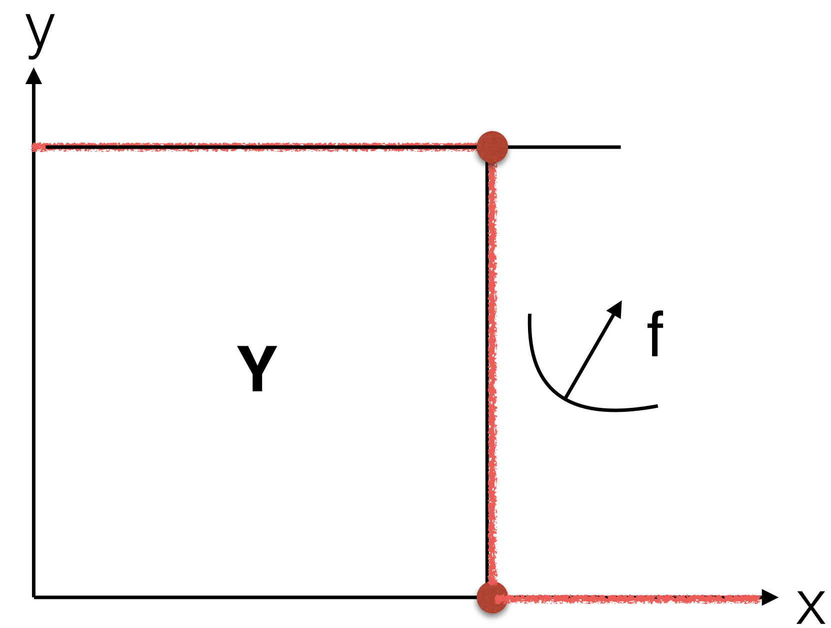
$$\begin{array}{ll} \max\limits_{x,y} & f_1 x + f_2 y \\ & \max\limits_{y} g_1 x + g_2 y \\ & \text{s.t.} (x,y) \in Y \cap X \end{array}$$

Multiple second level optima

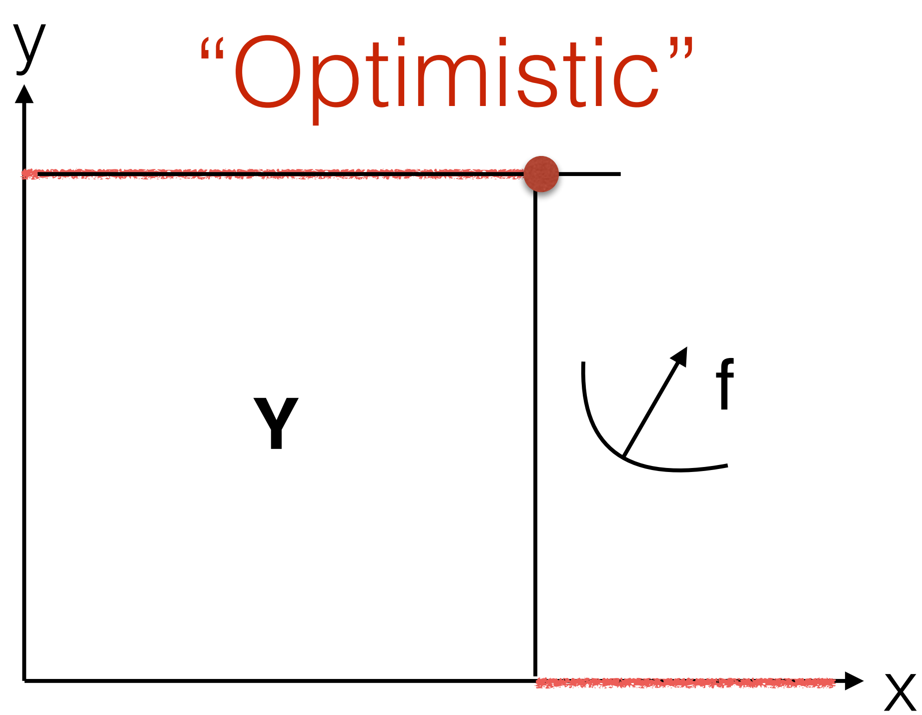
$$\max_{x \geq 0} \quad xy$$

$$\text{s.t.} \quad \max_{y} (1-x)y$$

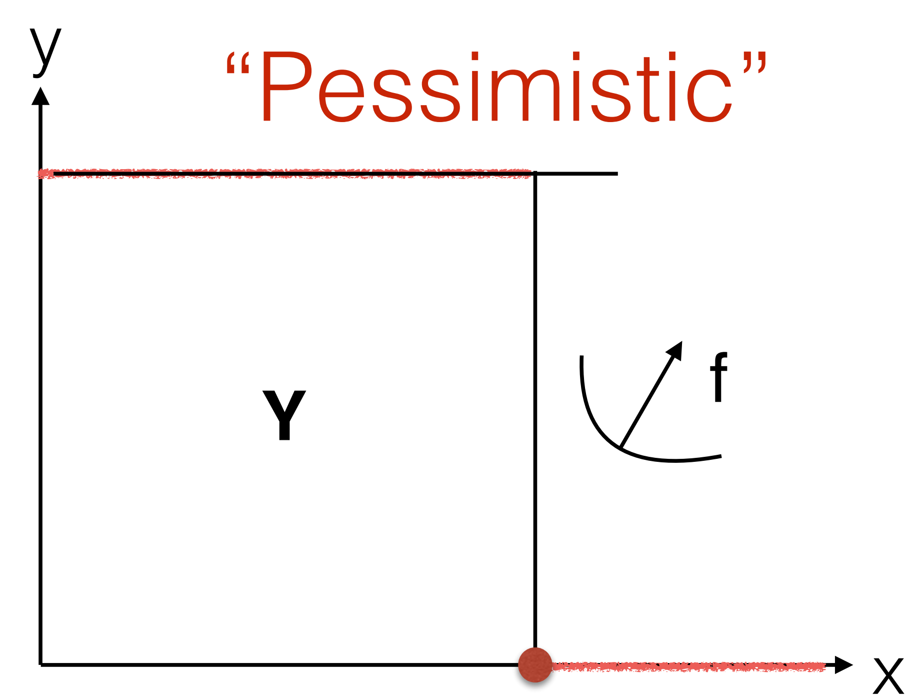
$$\text{s.t.} 0 \leq y \leq 1$$



“Optimistic”



“Pessimistic”



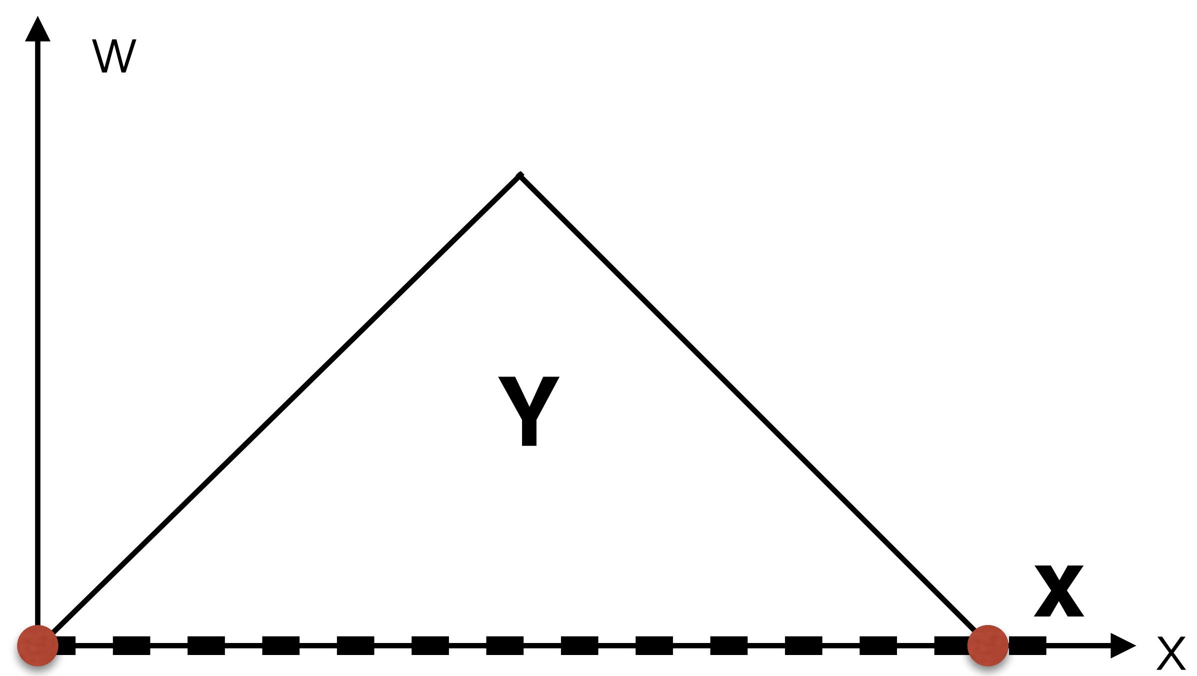
PART II:
Linear Bilevel optimization

Linear BP

$$\begin{aligned} \max_{x} \quad & c_1 x + d_1 y \\ \text{s.t.} \quad & A_1 x + A_1 y \leq b_1 \quad \}\mathbf{X} \\ & \max_{y} [c_2 x +] \, d_2 y, \\ & \text{s.t.} A_2 x + B_2 y \leq b_2 \quad \}\mathbf{Y} \end{aligned}$$

0/1 Programming is a special case of BO
(Audet et al. 1997)

$$\boxed{x \in \{0,1\} \Leftrightarrow v = 0 \text{ and } v = \underset{w}{\operatorname{argmax}}\{w : w \leq x, w \leq 1 - x, w \geq 0\}}$$



Linear BP

- Linear BP is strongly NP-hard (Hansen et al. 1992)

- MILP is a special case of Linear BP

- IR is not convex and may be disconnected.

Linear BP

(Bialas & Karwan(1982), Bard(1983)).

- IR is the union of faces of X∩Y

- If Linear BP is feasible, then there exists an optimal solution which is a vertex of X∩Y.

K-th best algorithm

Linear BP- single level reformulation

$$\begin{aligned}
\max_x \quad & c_1 x + d_1 y \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& \max_y d_2 y, \\
& \text{s.t.} \quad B_2 y \leq b_2 - A_2 x \qquad (\lambda)
\end{aligned}$$

$$\begin{aligned}
\max_x \quad & c_1 x + d_1 y \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& B_2 y \leq b_2 - A_2 x \\
& \lambda B_2 = d_2 \\
& \lambda \geq 0 \\
& \boxed{\lambda(B_2 y - b_2 + A_2 x) = 0}
\end{aligned}$$

Linear BP- single level reformulation

$$
\begin{aligned}
\max_{x} \quad & c_1 x + d_1 y \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& B_2 y + A_2 x \leq b_2 \\
& \lambda B_2 = d_2 \\
& \lambda \geq 0 \\
& \boxed{\lambda (B_2 y - b_2 + A_2 x) = 0}
\end{aligned}
$$

$$
\begin{aligned}
\max_{x} \quad & c_1 x + d_1 y \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& B_2 y + A_2 x \leq b_2 \\
& \lambda B_2 = d_2 \\
& \lambda \geq 0 \\
& \lambda \leq M_d z \\
& A_2 x + B_2 y \geq b_2 - M_p (1 - z) \\
& z \in \{0, 1\}^m
\end{aligned}
$$

Linear BP- single level reformulation

$$
\begin{aligned}
\max_x \quad & c_1 x + d_1 y \\
\text{s.t.} \quad & A_1 x + B_1 y \leq b_1 \\
& \max_y d_2 y, \\
& \text{s.t.} \quad B_2 y \leq b_2 - A_2 x \quad (\lambda)
\end{aligned}
$$

- Branch & Bound (Hansen et al.1992)
- Branch & Cut (Audet et al. 2007)
- Finding a valid M_d amounts finding a vertex of $\{\lambda B_2 = d_2, \lambda \geq 0\}$

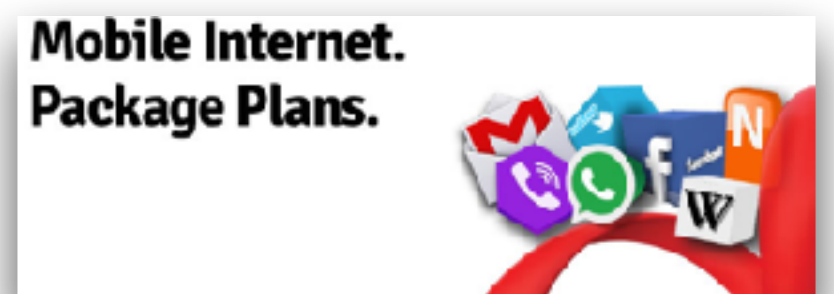
with largest coordinate which is NP-hard (Kleinert et al., 2019).

PART III:
Pricing problems

Adequate framework for Price Setting Problem

$$\boxed{\begin{array}{ll} \max\limits_{T \in \Theta, x, y} & F(T, x, y) \\ \text{s.t.} & \min\limits_{x, y} f(T, x, y) \\ & \text{s.t.} (x, y) \in \Pi \end{array}}$$

Applications



Price Setting Problem with linear constraints

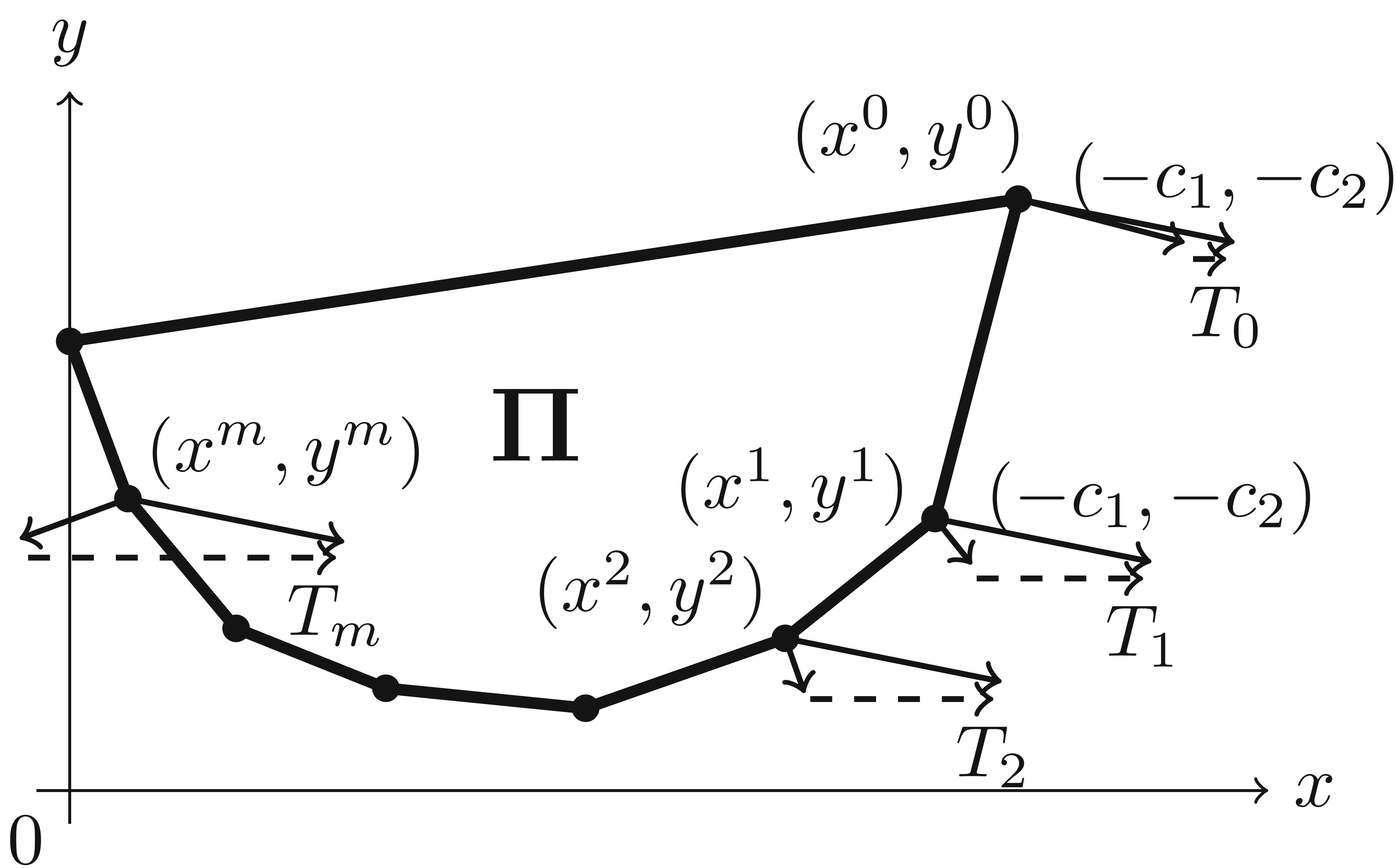
$$\begin{aligned} \max_{T,x,y} \quad & Tx \\ \text{s.t.} \quad & TC \geq f \\ \min_{x,y} \quad & (c+T)x + dy \\ \text{s.t.} \quad & Ax + By \geq b \end{aligned}$$

- $\Pi = \{x, y : Ax + By \geq b\}$ is bounded
- $\{(x, y) \in \Pi : x = 0\}$ is nonempty

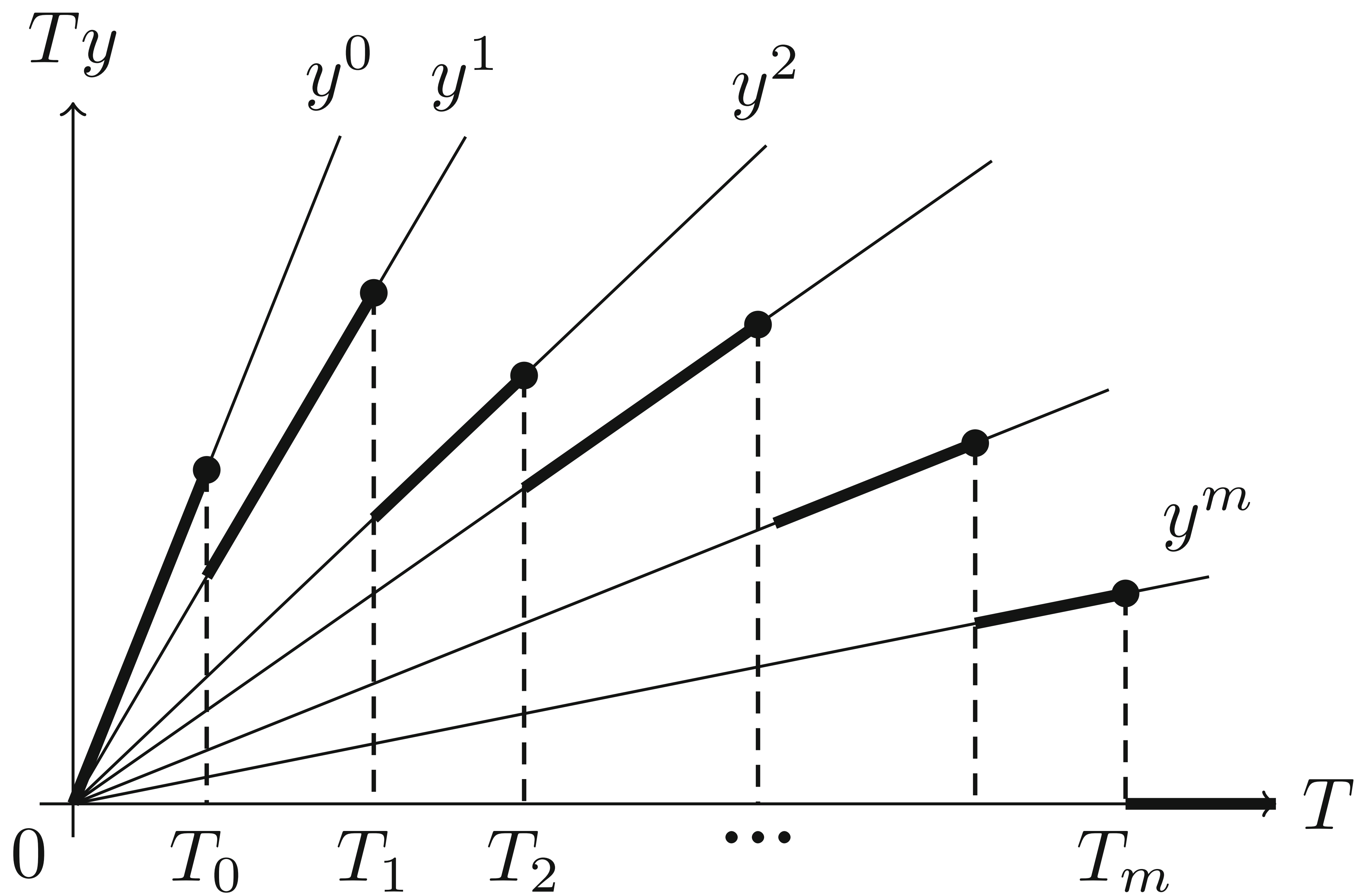
Example: 2 variables in second level

$$\begin{aligned} \max_{T,x,y} \quad & Ty \\ \text{s.t.} \min_{x,y} \quad & c_1 x + (c_2 + T)y \\ \text{s.t.} \quad & (x, y) \in \Pi \end{aligned}$$

Example: 2 variables in second level



The first level revenue

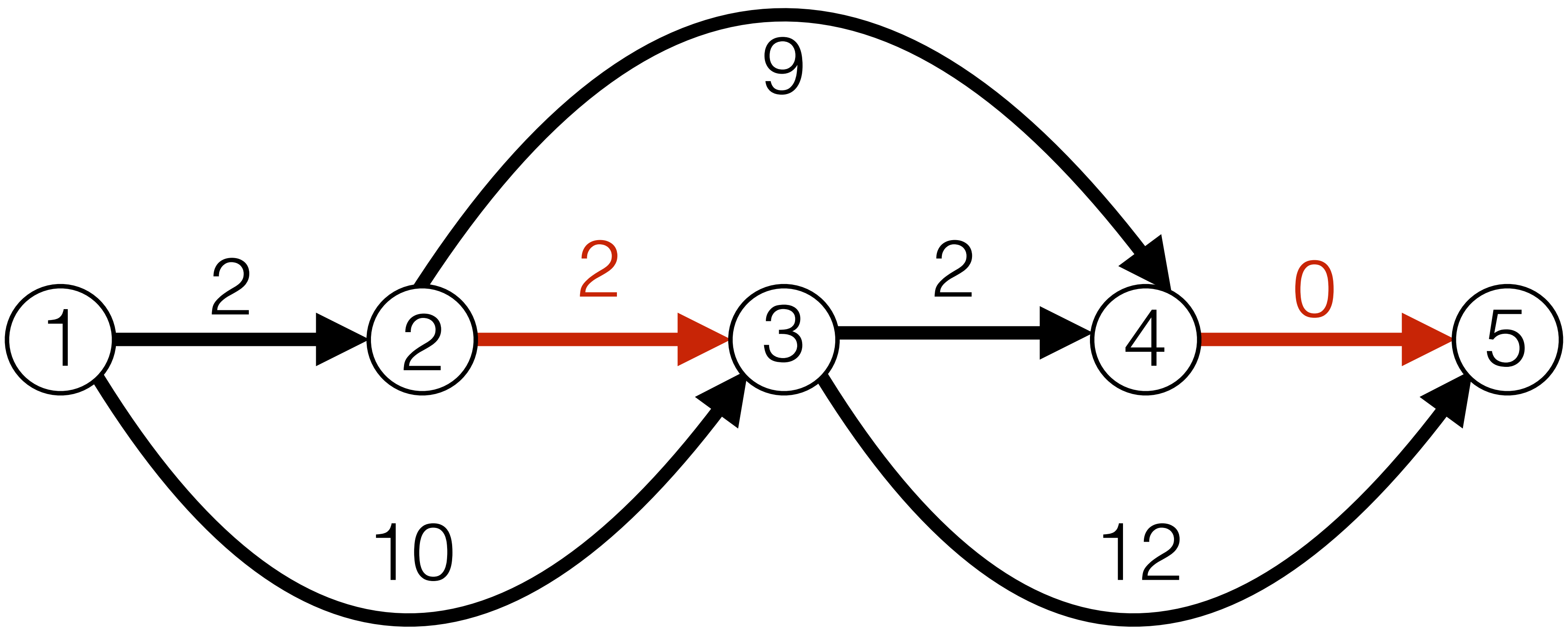


Network pricing problem

(Labbé et al.1998)

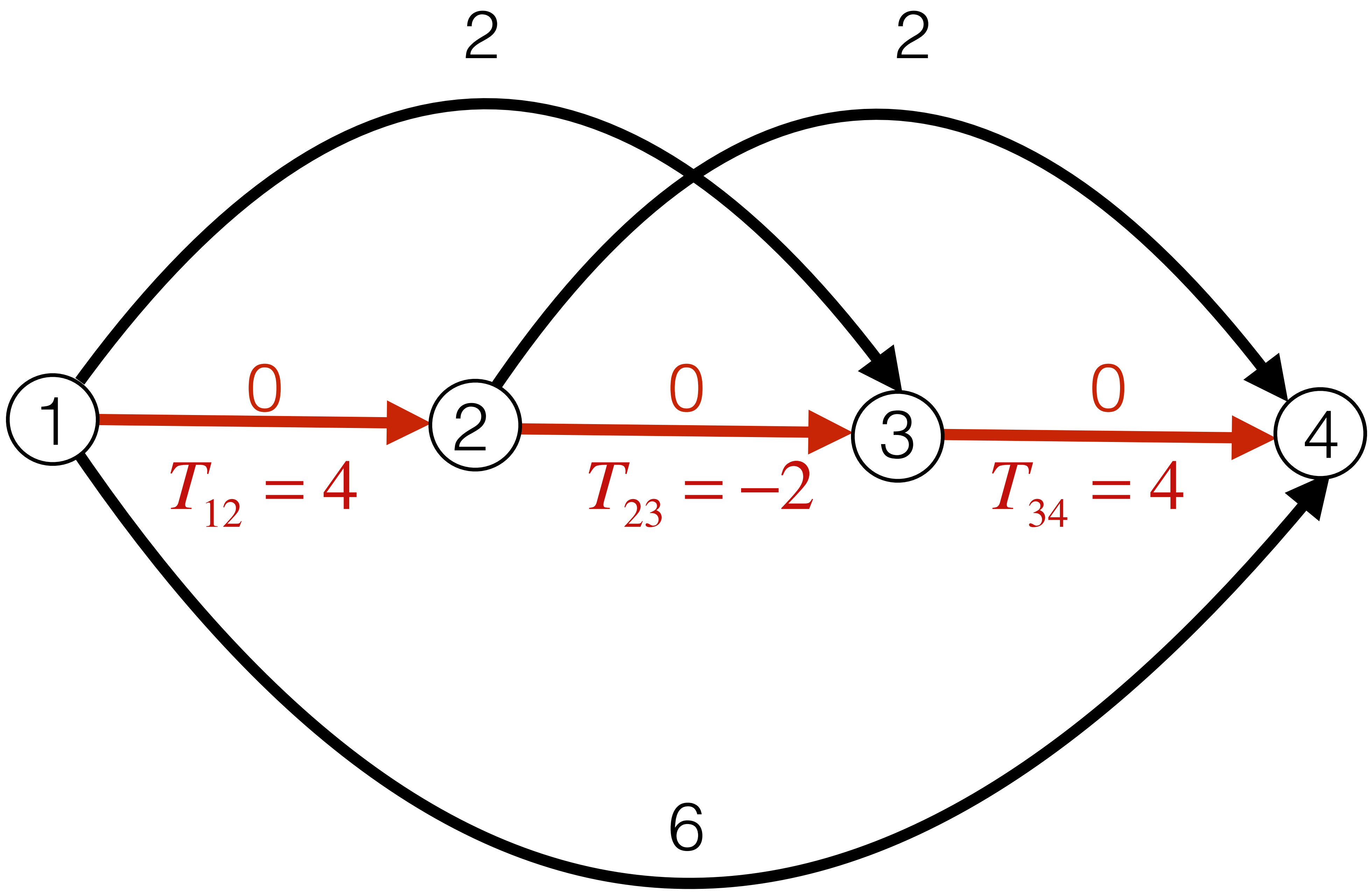
- network with toll arcs (A_1) and non toll arcs (A_2)
- Costs c_a on arcs
- Commodities (o^k, d^k, n^k)
- Routing on cheapest (cost + toll) path
- Maximize total revenue

Example



- $\text{UB on } (T_1 + T_2) = \text{SPL}(T = \infty) - \text{SPL}(T = 0) = 22 - 6 = 16$
- $T_{2,3} = 5, T_{4,5} = 10$

Example with negative toll arc



Network pricing problem

$$
\begin{aligned}
\max_{T \geq 0} \quad & \sum_{a \in A_1} T_a \sum_{k \in K} n^k x_a^k \\
\min_{x,y} \quad & \sum_{k \in K} \Big(\sum_{a \in A_1} (c_a + T_a) x_a^k + \sum_{a \in A_2} c_a y_a \Big) \\
\text{s.t.} \quad & \sum_{a \in i^+} (x_a^k + y_a^k) - \sum_{a \in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i \\
& x_a^k, y_a^k \geq 0, \quad \forall k, a
\end{aligned}
$$

NPP: single level reformulation

$$\max_{T,x,y,\lambda} \quad \sum_{k\in K} n^k \sum_{a\in A_k} T_a x_a^k$$

$$\text{s.t.} \quad \sum_{a\in i^+} (x_a^k + y_a^k) - \sum_{a\in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i$$

$$\lambda_i^k - \lambda_j^k \leq c_a + T_a \quad \forall k, a \in A_1, i, j$$

$$\lambda_i^k - \lambda_j^k \leq c_a \quad \forall k, a \in A_2, i, j$$

$$\sum_{a\in A_1} (c_a + T_a) x_a^k + \sum_{a\in A_2} c_a y_a = \lambda_{o^k}^k - \lambda_{d^k}^k \quad \forall k$$

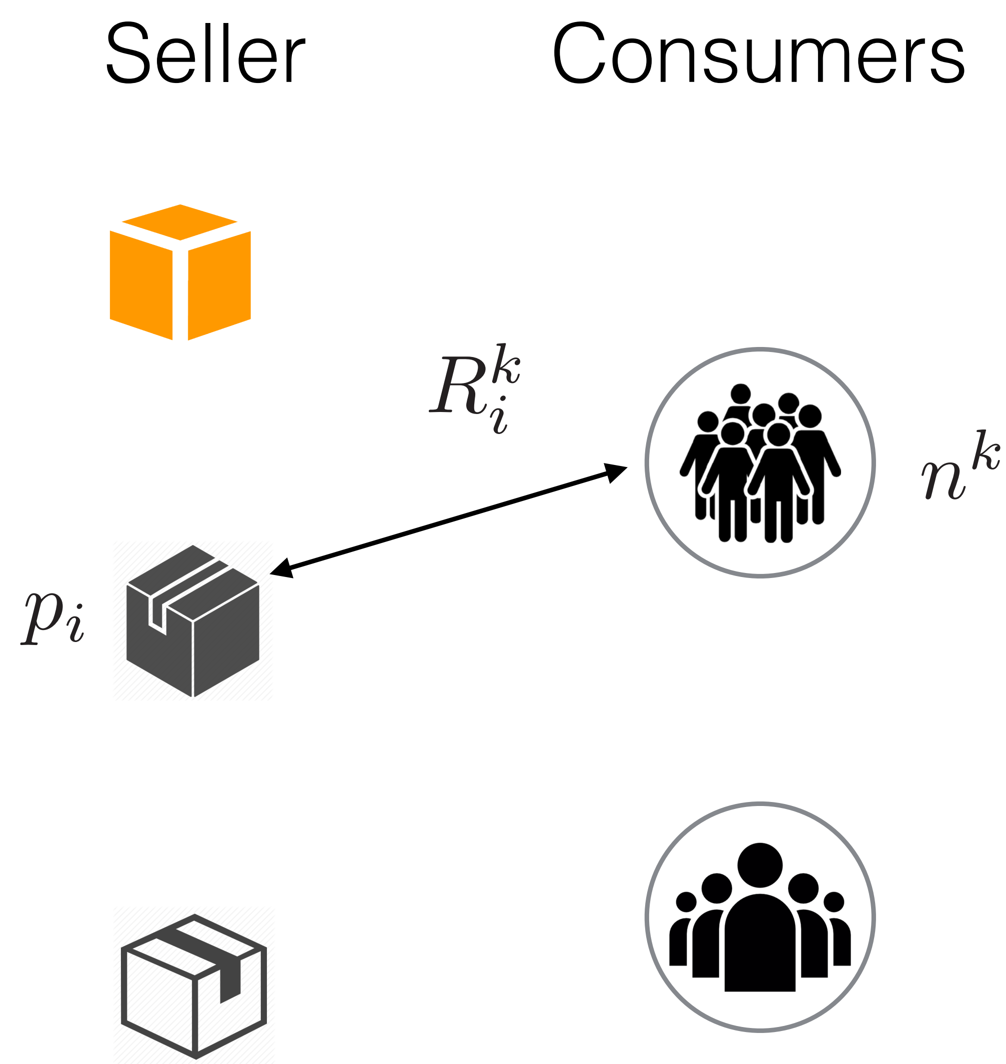
$$x_a^k, y_a^k \geq 0 \quad \forall k, a$$

$$T_a \geq 0 \quad \forall a \in A_1$$

Solution approach

- MILP formulation
- Tight bound “big M” very effective
- Branch & cut

Product pricing



R_i^k is the reservation price of consumer k for product i

Product pricing

- PPP is Strongly NP-hard even if reservation price is independent of product (Briest 2006)

- PPP is polynomial for one product or one customer.

PPP - bilevel formulation

$$
\begin{aligned}
\max_{p \geq 0} \quad & \sum_{k \in K} n^k \sum_{i \in I} p_i x_i^k \\
\text{s.t.} \quad & \max_{x^k} \sum_{i \in I} (R_i^k - p_i) x_i^k, \quad k \in K \\
& \text{s.t.} \quad \sum_{i \in I} x_i^k \leq 1 \\
& \qquad\quad x_i^k \geq 0
\end{aligned}
$$

PPP - single level formulation

$$
\begin{aligned}
\max_{p \geq 0} \quad & \sum_{k \in K} n^k \sum_{i \in I} p_i x_i^k \\
\text{s.t.} \quad & \sum_{i \in I} (R_i^k - p_i) x_i^k \geq R_j^k - p_j, \quad j \in I, k \in K \\
& \sum_{i \in I} (R_i^k - p_i) x_i^k \geq 0, \quad k \in K \\
& \sum_{i \in I} x_i^k \leq 1 \\
& x_i^k \geq 0
\end{aligned}
$$

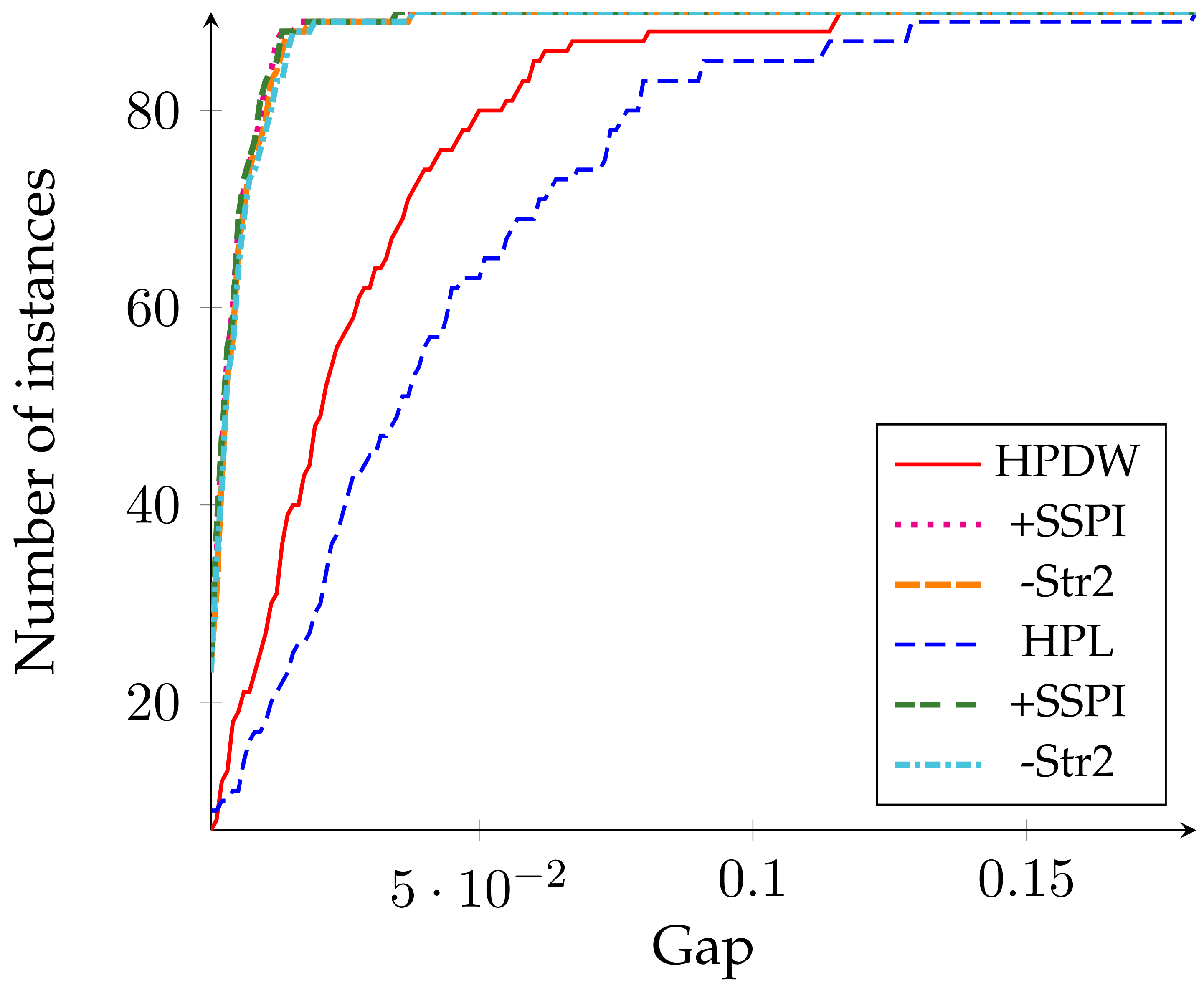
PPP: MILP formulation

(Heilporn et al., 2010, 2011)

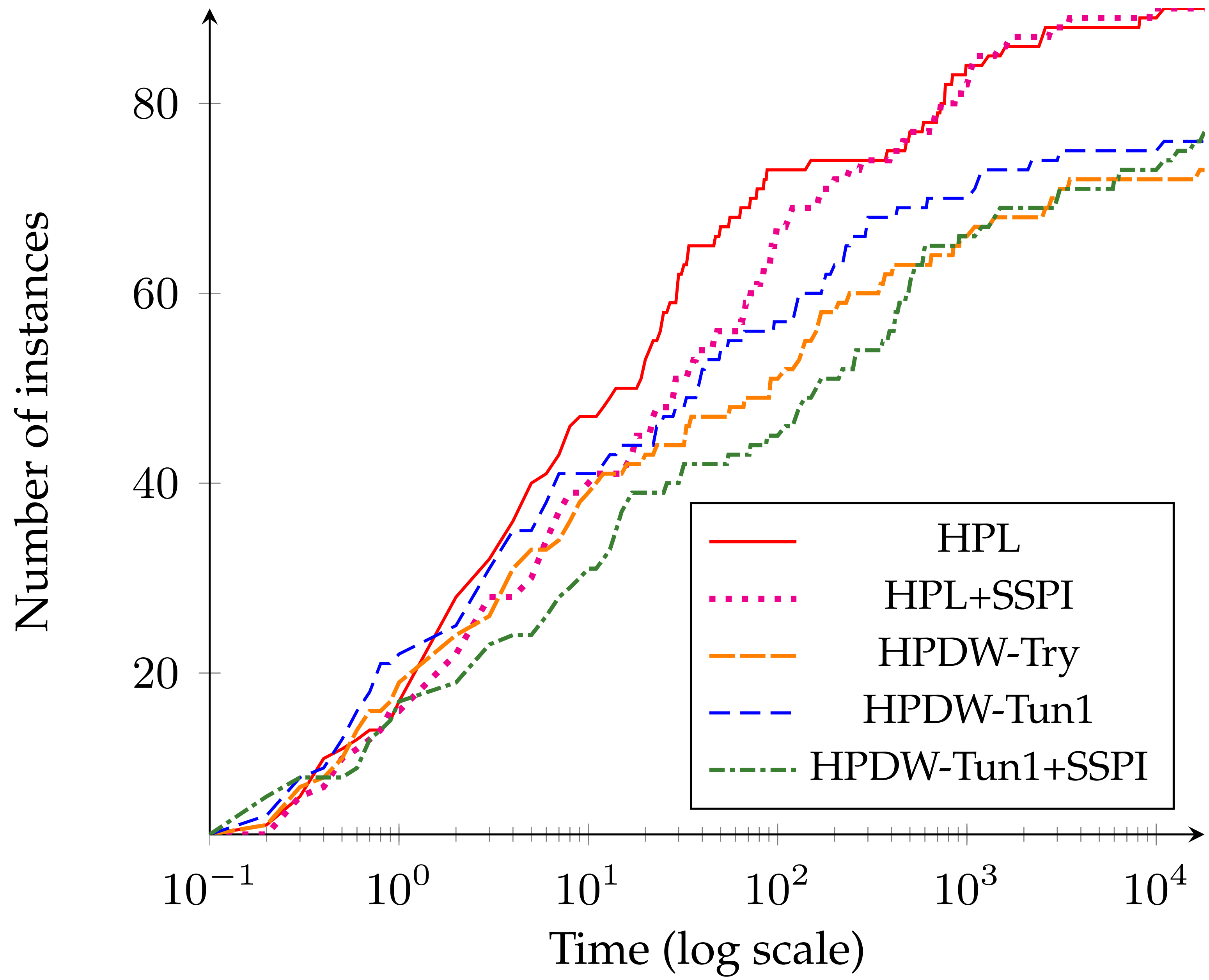
- MILP formulations
- Convex hull for k=1
- Branch & cut, branch & price

PPP: gap
(Violin, 2014)

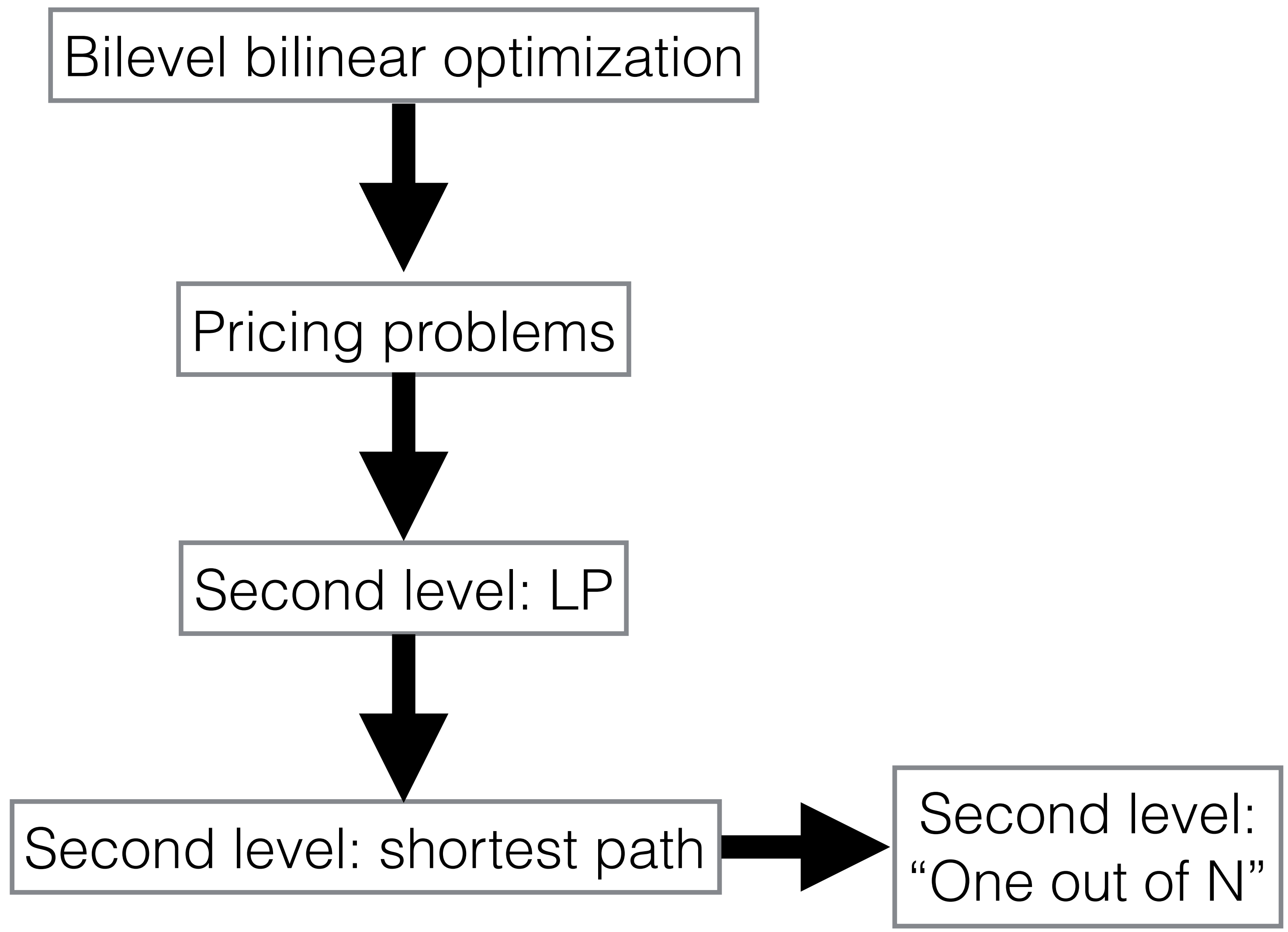
20 - 90 products
20 - 90 customers



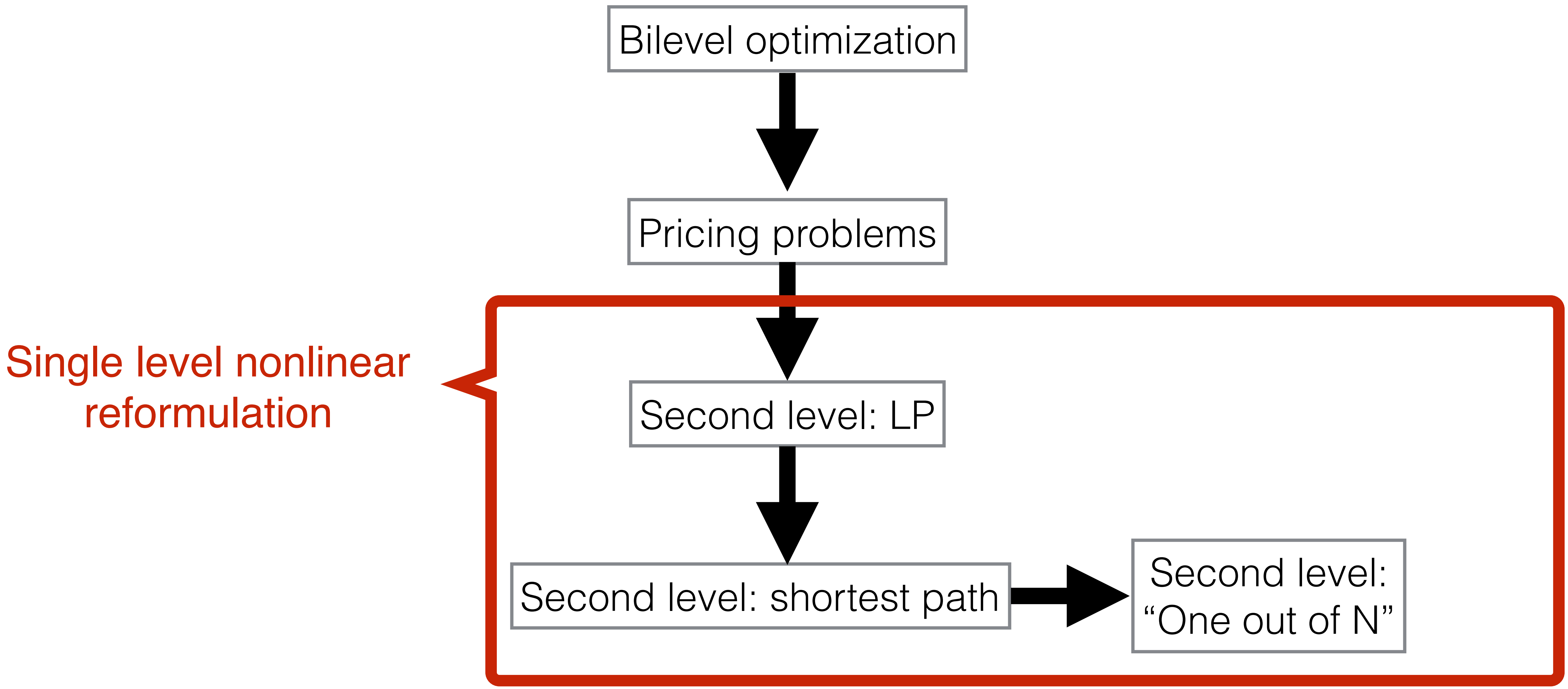
PPP: computing time



RECAP

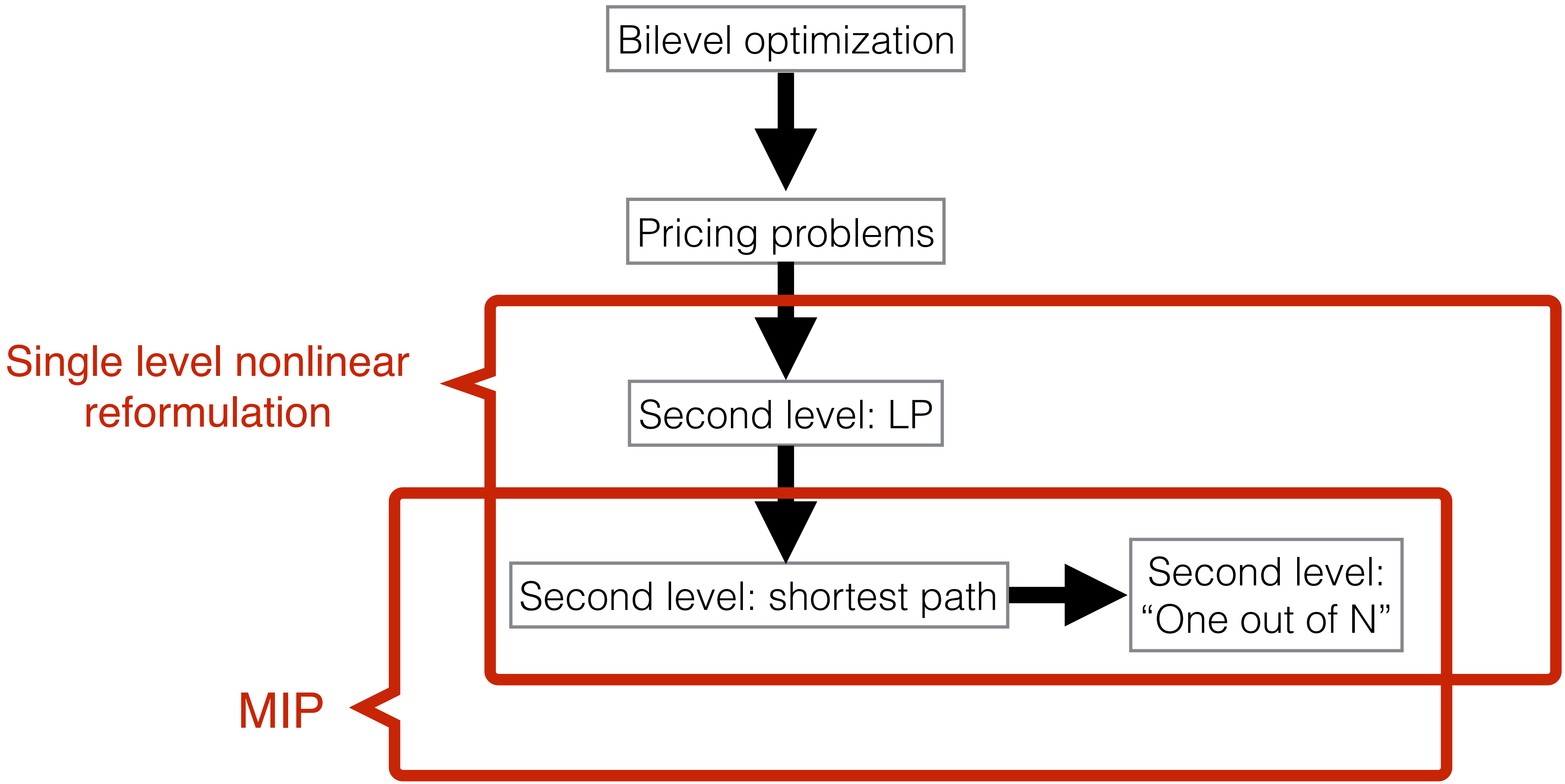

Bilevel bilinear optimization
Pricing problems
Second level: LP
Second level: shortest path
Second level:
“One out of N”

RECAP


Bilevel optimization
Pricing problems
Single level nonlinear reformulation
Second level: LP
Second level: shortest path
Second level: “One out of N”

42

RECAP


Bilevel optimization
Pricing problems
Single level nonlinear reformulation
Second level: LP
Second level: shortest path
Second level: “One out of N”
MIP

43

Conclusion

- Bilevel model: rich framework for pricing in network-based industries.
- Models: theoretically and computationally challenging.
- Need to exploit problem's inner structure.
- Analysis of basic model: relevant and useful for attacking real applications (http://www.expretio.com/).
- Integration of real-life features (congestion, market segmentation, dynamics, randomness...).
- Investigate variants of product pricing.


Thank You
Merci
Arigato
Grazie
Gracias
Danke
Efharisto
Hvala
Spasiba
Cheers
Toda
Obrigado
Obrigada
Bedankt
Barkal
Multumesc
Blagodaram
Kiitos
Dziekuje
Shukran
Meharbani
Salamat
Nandri
Dekuji
Manjuthe
Rahmat
Gràcies
Merkzi
Murakoze
Chokrane
Shakkran
Akiba
Aabar
Dakujem
Mèsi
Nkosi
Tack
Tanmirt
Buznyg
Danki
Aitäh
Bayarlalaa
Ahsante
Tashakor
Yekeniele
Cheers
Dziakuju
Paldies
Ngiyabonga
Manana
Dankegon
Mahalo
Madlobt
Misaotra
Gratzias
Takk
Nuhun
Matondo
Blagodaria
Sobodi
Dankewol
Skee
Trugéré
Shukria
Tenki
Wado
Grassie
Modupe
Omol
Sağol
Vinaka
Menlau
Köszönöm
Marahaba
Mauruuru
Soolong
Welalin
Saha
Spas

Some references

- L.N. Vincente and P.H. Calamai (1994), Bilevel and multilevel programming : a bibiliography review, J. Global Optim. 5, 291-306.

- S. Dempe. Foundations of bilevel programming. In Nonconvex optimization and its applications, volume 61. Kluwer Academic Publishers, 2002.

- J. Bard. Practical Bilevel Optimisation: Algorithms and Applications. KluwerAcademic Publishers, 1998

- B. Colson, P. Marcotte, and G. Savard. Bilevel programming: A survey. 4 OR, 3:87-107, 2005.

- M. Labbé and A. Violin. Bilevel programming and price setting problems. 4OR, 11:1-30, 2013.

Editorial Board

EURO Journal on Computational Optimization – 2014 Publisher's Report